On the minimum distance of AG codes, on Weierstrass semigroups and the smoothability of certain monomial curves in 4-Space.

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Abstract

In this paper we treat several topics regarding numerical Weierstrass semigroups and the theory of Algebraic Geometric Codes associated to a pair (X, P), where X is a projective curve defined over the algebraic closure of the finite field $\mathbb{F}_q$ and P is a $\mathbb{F}_q$-rational point of X. First we show how to evaluate the Feng-Rao Order Bound, which is a good estimation for the minimum distance of such codes. This bound is related to the classical Weierstrass semigroup of the curve X at P. Further we focus our attention on the question to recognize the Weierstrass semigroups over fields of characteristic 0. After surveying the main tools (deformations and smoothability of monomial curves) we prove that the semigroups of embedding dimension four generated by an arithmetic sequence are Weierstrass.

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0 Introduction.

The paper is divided into two parts. In the first one we describe some bounds of the minimum distance of AG codes, while in the second one we deal with the problem to characterize the Weierstrass semigroups.

In the first part $\mathbb{F}$ will denote the algebraic closure of the finite field with q elements $\mathbb{F}_q$; X will be a smooth projective algebraic curve of genus g defined over $\mathbb{F}_q$.
To a pair (X, P), where $P \in X$ is a $\mathbb{F}_q$-rational point can be associated a family of *Algebraic Geometric Codes* $C_i,\ i \in \mathbb{N}$ and a *numerical semigroup S*. For i large enough, the minimum distance $d(C_i)$ of such codes can be bounded by the Feng-Rao order bound $d_{ord}(C_i)$ which depends only on the semigroup S (see [10]). When S is non-ordinary, it is called the Weierstrass semigroup of X at P. Evaluations or estimates of the order bound are given by several authors, either in general or in particular cases (see, e.g., [1], [22]). In the first part of this paper we give a survey of these results and we state a conjecture (2.3) on the behaviour of the sequence $\{d_{ord}(C_i)\}_{i\in\mathbb{N}}$, for $i > c + d - e - g$, where c, d, e are suitable integers associated to the semigroup S (in [22] this conjecture is proved in many cases).

According to the recalled relation with code theory, the classical study of Weierstrass semigroups is becoming relevant. In particular an interesting and still open hard question is how to

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recognize Weierstrass semigroups, i.e. those semigroups associated to a smooth projective curve at a point P. This problem is approached in the second part under the simplifying assumption that X is a smooth projective algebraic curve of genus g defined over an algebraically closed field $\mathbb{F}$ with char$(\mathbb{F}) = 0$. It is known that there exist non-Weierstrass semigroups: the first example is due to Buchweitz, see [2]. A fundamental result on this topic has been proved by Pinkham in his Phd thesis [25]:

"S is Weierstrass if and only if the monomial curve $X =$ Spec$(\mathbb{F}[S])$ is smoothable".

In some case it is known that a monomial curve is smoothable: see [27] for the complete intersection case, see [29] for $X \subseteq \mathbb{A}^3$, see [16] for $X \subseteq \mathbb{A}^4$ and Gorenstein, see [18] for semigroups of genus $g \leq 8$, see [17], [19] for certain semigroups of embedding dimension 5 or with $g = 9$.

In this paper we collect the main definitions and results on this question, further we illustrate the explicit algorithm to obtain a deformation of a monomial X with its $\mathbb{G}_m$ action and show several examples in a detailed way. Finally we show that monomial curves in $\mathbb{A}^4$, generated by an arithmetic sequence are smoothable. It follows that every semigroup S of embedding dimension 4 generated by an arithmetic sequence is Weierstrass.

1 Weierstrass points and Weierstrass semigroups

Let $\mathbb{F}$ denote an algebraically closed field. Let X be a smooth projective algebraic curve of genus g defined over $\mathbb{F}$ with function field $\mathbb{F}(X)$, and let $P \in X$. For each $k \in \mathbb{N}$, let

$$\mathcal{L}(kP) = \{f \in \mathbb{F}(X) \setminus 0 \mid \text{div}(f) + kP \geq 0\} \cup \{0\}.$$

This is clearly a vector subspace of $\mathbb{F}(X)$; we denote by $\lambda(kP)$ its dimension over $\mathbb{F}$. The following are well-known facts:

$\lambda(kP) = \dim_{\mathbb{F}}(\mathcal{L}(kP)) \in \mathbb{N}$, $\quad \lambda((k-1)P) \leq \lambda(kP) \leq \lambda((k-1)P) + 1$ for each $k > 1$, and by Riemann-Roch Theorem $\lambda(nP) = n - g + 1$ for each $n \geq 2g - 1$.

Hence the set $H(P) := \{k \in \mathbb{N}_+ \mid \lambda((k-1)P) = \lambda(kP)\}$, of *gaps at P*, is a proper subset of $\{1, 2, \ldots, 2g-1\}$ and it has exactly g elements. Moreover it is easy to see that its complement $S(P) := \mathbb{N} \setminus H(P)$, the *set of non-gaps at P*, is a numerical semigroup.

Recall that a semigroup S is called *ordinary* if it is of the form $S = \{0, e, e+1 \rightarrow\}$ for some $e > 0$ (note that its *genus*, also called δ, is exactly $e - 1$).

Definition 1.1 *A Weierstrass point of X is a point P such that $H(P) \neq \{1, \ldots, g\}$. A semigroup S is called Weierstrass (over $\mathbb{F}$) if there exists a smooth projective algebraic curve X (defined over $\mathbb{F}$) and a Weierstrass point P such that $S = S(P)$.*

See for more details, e.g., [12, Exercise A.4.14] or [6].

Remark 1.2 Let $P \in X$, then by Riemann-Roch Theorem

1. $n \in S(P) \iff$ there exists $f \in \mathbb{F}(X)$ such that $(f)_\infty = nP$, i.e. ord$_P(f) = -n$.
2. $n \in H(P) \iff$ there exists a regular differential form ω with ord$_P(\omega) = n - 1$ (because by Riemann-Roch theorem: $\lambda(K - (n-1)P) > 0$, for each *gap* $n \in H$, where K denotes a canonical divisor).

3. P is a Weierstrass point $\iff \lambda(gP) \geq 2 \iff$ there exists a regular differential form ω with $\text{ord}_P(\omega) \geq g$. In particular, it follows immediately that if X has a Weierstrass point then $g \geq 2$.

4. By the previous point, the presence of a Weierstrass point on an algebraic curve of genus g ensures the existence of a morphism of degree not exceeding g from the curve onto the projective line: pick the morphism associated to the linear system $|iP|$ with any i such that $\lambda(iP) = 2$ and $i \leq g$.

1.1 On the number of Weierstrass points on a curve.

For a smooth curve X let W denote the set of Weierstrass points of X. We know that

1. If $g \leq 1$ the set W is empty.

2. Case X *hyperelliptic*. A hyperelliptic curve is an algebraic curve which admits a double cover over $I\!P^1$. These curves are among the simplest algebraic curves: they are all birationally equivalent to curves given by an equation of the form $y^2 = f(x)$ in the affine plane, where $f(x)$ is a polynomial of degree > 4 with distinct roots, and the degree of $f(x)$ is either twice the genus of the curve plus 2, or twice the genus of the curve plus one.

   If a double cover exists, then it is the unique double cover and it is called the "hyperelliptic double cover". In algebraic geometry the Riemann-Hurwitz formula, states that if X, X' are smooth algebraic curves, and $\Phi : X \longrightarrow X'$ is a finite map of degree d then the number of branch points of Φ, denoted by N, is given by

$$2g(X) - 2 = 2d(g(X') - 1) + N$$

   By the Riemann-Hurwitz formula the hyperelliptic double cover has $X' = I\!P^1$, hence has exactly $2g + 2$ branch points. For each branch point P we have $\lambda(2P) = 2$, hence these points are all Weierstrass points; for each of them there exists a function f with a double pole at P only. Its powers have poles of order $4, 6$, and so on. Therefore at P the gap sequence is $1, 3, 5, ..., 2g - 1$ and $\lambda(kP) = 2k$, we conclude that the Weierstrass points of X are exactly the $2g + 2$ branch points of the hyperelliptic double cover.

3. For algebraic curves of genus g there always exist at least $2g + 2$ Weierstrass points and only the hyperelliptic curves of genus g have exactly $2g + 2$ Weierstrass points.

4. The upper bound on the number of Weierstrass points is $g^3 - g$.

5. [14] For each $g \geq 3$ there exist compact Riemann surfaces of genus g with at least two Weierstrass point with different gap sequences.

2 Algebraic-geometric codes

Let now $\mathbb{F}$ denote an algebraic closure of the finite field with p elements $\mathbb{F}_p$, p prime. Let X be a smooth projective algebraic curve of genus g defined over $\mathbb{F}_q$, $q = p^r$ for some $r \in \mathbb{N}_+$,

with function field $\mathbb{F}(X)$. Let $P \in X$ be an $\mathbb{F}_q$-rational point: a family of *codes* and a *numerical semigroup* can be associated to (X, P) as follows. For each $k \in \mathbb{N}$, we consider the vector subspace of $\mathbb{F}_q(X)$ defined as

$$\mathcal{L}_{\mathbb{F}_q}(kP) = \{f \in \mathbb{F}_q(X) \setminus 0 \mid \text{div}(f) + kP \geq 0\} \cup \{0\},$$

it can be shown that $\lambda(kP) = \dim_{\mathbb{F}}(\mathcal{L}(kP)) = \dim_{\mathbb{F}_q}(\mathcal{L}_{\mathbb{F}_q}(kP))$ ([12, Proposition A.2.2.10.]).

We now recall the definitions of the AG codes associated to the pair (X, P). Choose $P_1, ..., P_n$ distinct $\mathbb{F}_q$-rational points on X such that $P_j \neq P$ for each j, and consider the $\mathbb{F}_q$-linear map

$$\Phi_k \colon \mathcal{L}_{\mathbb{F}_q}(kP) \longrightarrow {\mathbb{F}_q}^n \quad \text{as} \quad \Phi_k(f) = (f(P_1), ..., f(P_n)).$$

Definition 2.1 *The family of one-point AG codes of order n is defined as*

$$C_k := (Im\ \Phi_k)^{\perp} = \{x \in {\mathbb{F}_q}^n \mid < x, \Phi_k(f) >= 0 \textit{ for all } f \in \mathcal{L}_{\mathbb{F}_q}(kP)\},$$

where $< x, y >:= x_1 y_1 + \cdots + x_n y_n$ *for each* $x, y \in {\mathbb{F}_q}^n$*.*

A good estimate of the *minimum distance* $d(C_k)$ of an AG code is the Feng-Rao order bound $d_{ORD}(C_k)$ which depends only on the semigroup $S = S(P)$. Let us fix the following notation

$$S = \{s_0 = 0, s_1, ..., s_j, ...\} \neq \mathbb{N}$$

with $s_i < s_j$ if $i < j$.

Definition 2.2 *For $s_j \in S$, let* $\left[\begin{array}{lcl} N(s_j) & := & \{(s_h, s_k) \in S^2 \mid s_j = s_h + s_k\} \\ \nu(s_j) & := & \#N(s_j) \end{array} \right.$.

The Feng-Rao order bound of the code C_k is $d_{ORD}(C_k) \ := \ \textit{min}\{\nu(s_j) \mid j > k\} \leq d(C_k).$

If S is ordinary, that is $S = \{s_0 = 0, s_1 = g+1, s_2 = g+2 \rightarrow\}$, the sequence $\{\nu(s_j),\ j \in \mathbb{N}\}$ is non-decreasing and so

$$d_{ORD}(C_k) = \nu(s_{k+1}) \ \ for \ \ k \geq 0.$$

In the other cases, it is known that there exists $m \in \mathbb{N}_+$ such that

$$\nu(s_m) > \nu(s_{m+1}) \ \ and \ \ \nu(s_{m+i}) \leq \nu(s_{m+i+1}) \ \ \forall\, i \geq 1.$$

Then: $d_{ORD}(C_k) = \nu(s_{k+1})$ for each code C_k with $k \geq m$.

2.1 Methods for the evaluation of s_m.

Our goal is to find s_m for a given semigroup S; to this end it is useful to consider the elements of S "near" the conductor.

Notation 2.3 *We shall refer to a numerical semigroup S, with finite complement in $\mathbb{N}$*

$$S = \{0 = s_0, s_1, ..., s_j, ...\} \neq \mathbb{N}$$

where $s_i < s_k$, if $i < k$. Further we denote:

$$\begin{aligned}
&\textit{embdim}(S) \;=\; minimal\ number\ of\ generators\ of\ S \\
&e = s_1 = \textit{min}\{s \in S \mid s \neq 0\},\ the\ multiplicity \\
&c = \textit{min}\{r \in S \mid r + \mathbb{N} \subseteq S\},\ the\ conductor \\
&d = \textit{max}\{s_i \in S \mid s_i < c\},\ the\ dominant \\
&c' = \textit{max}\{s_i \in S \mid s_i \leq d\ \ and\ \ s_i - 1 \notin S\},\ the\ subconductor \\
&d' = the\ greatest\ element\ in\ S\ preceding\ c',\ when\ \ c' > 0 \\
&\ell = c - 1 - d,\ the\ number\ of\ gaps\ of\ S\ greater\ than\ d \\
&\widetilde{s} = \textit{max}\{s \in S, \mid s \leq d,\ \ s - \ell \notin S\}.
\end{aligned}$$

This means that S has the following shape (thinking of it as embedded in $\mathbb{N}$, where $*$ means a "gap" of S)

$$S = \{0,\ \ \overset{e-1 \text{ gaps}}{* \cdots *}\ \ \ e, \ldots, d',\ \ \overset{c'-d'-1 \text{ gaps}}{* \cdots *}\ \ \ c' \longleftrightarrow d,\ \ \overset{\ell \text{ gaps}}{* \cdots *}\ \ \ c \rightarrow\}$$

A semigroup S is called *acute* if either S is ordinary, or c, d, c', d' satisfy $c - d \leq c' - d'$ (see [1]). If S is non-ordinary, it can be seen that:

$$S \text{ acute} \implies c' \leq \widetilde{s} \leq d.$$

Example 2.4 $S = \{0, 8_e, 12_{d'}, 14_{c'}, 15, 16_d, 20_c \rightarrow\}$ *has* ($\ell = 3,\ \widetilde{s} = 14,\ c' - d' = 2 < c - d = 4$, S *non-acute*).

From now on, S will be non-ordinary. In order to evaluate s_m we study the difference $\nu(s_{i+1}) - \nu(s_i)$ for $s_i \in S$. To this end, it is "natural" to consider the following partition of $N(s_i) = \{(s_j, s_k) \in S^2 \mid s_i = s_j + s_k\}$:

$$N(s_i) = A(s_i) \cup B(s_i) \cup C(s_i) \cup D(s_i)$$

$$\begin{aligned}
A(s_i) &:= \{(x, y), (y, x) \in N(s_i) \mid x < c',\ c' \leq y \leq d\} \\
B(s_i) &:= \{(x, y) \in N(s_i) \mid (x, y) \in [c', d]^2\ \} \\
C(s_i) &:= \{(x, y) \in N(s_i) \mid\ x \leq d',\ y \leq d'\} \\
D(s_i) &:= \{(x, y),\ (y, x) \in N(s_i) \mid\ x \geq c,\ x \geq y\}.
\end{aligned}$$

Example 2.5 $S = \{0, 8_e, 12_{d'}, 14_{c'}, 15, 16_d, 20_c \rightarrow\}$. *For* $i = 16,\ s_i = 30:$
$A(s_i) = C(s_i) = \emptyset$, $B(s_i) = \{(14, 16), (15, 15), (16, 14)\}$,
$D(s_i) = \{(0, 30), (8, 22), (30, 0), (22, 8)\}$.
For $s_6 = 20:\ \ A(s_6) = B(s_6) = \emptyset,\ C(s_6) = \{(8, 12), (12, 8)\}\ ,\ \ D(s_6) = \{(0, 20), (20, 0)\}$.

Setting 2.6

$$\begin{aligned} \alpha(s_i) &:= \#A(s_{i+1}) - \#A(s_i) \\ \beta(s_i) &:= \#B(s_{i+1}) - \#B(s_i) \\ \gamma(s_i) &:= \#C(s_{i+1}) - \#C(s_i) \\ \delta(s_i) &:= \#D(s_{i+1}) - \#D(s_i). \end{aligned}$$

Therefore: $\nu(s_{i+1}) - \nu(s_i) = \alpha(s_i) + \beta(s_i) + \gamma(s_i) + \delta(s_i)$.

Lemma 2.7 (*see* [22])

1. $\alpha(s_i) \in \{-2, 0, 2\}$ *and* $\alpha(s_i) = 0$, *if* $s_i > d' + d$.

2. $\beta(s_i) \in \{-1, 0, 1\}$ *and* $\beta(s_i) = 0$, *if* $s_i > 2d$.

3. $\gamma(s_i)$ is difficult to evaluate if $s_i < 2d'$, trivial otherwise:

   in fact $\gamma(2d') = -1$ and $\gamma(s_i) = 0$, if $s_i > 2d'$.

4. *If* $s_i \geq 2c$, *then* $\delta(s_i) = 1$.

   If $s_i < 2c$ *and* $s_{i+1} \in S$, *then* $\delta(s_i) \in \{0, 2\}$ *and* $\delta(s_i) = 0 \Longleftrightarrow s_{i+1} - c \notin S$.

5. $s_m \leq 2d$. (In fact by (1)-(4), if $s_i \geq 2d + 1$ then $\alpha = \beta = \gamma = 0$ and so $\nu(s_{i+1}) - \nu(s_i) = \delta(s_i) \geq 0$.

By (2.7.5), from now one has to consider only elements $s_i \leq 2d$, in order to find the greatest $s_i \in S$ such that $\nu(s_{i+1}) < \nu(s_i)$. Assume $s_i + 1 \in S$.

Remark 2.8 1. *If* $s_i = \widetilde{s} + d$, *then* $s_{i+1} - c = \widetilde{s} - \ell \notin S$ *(by definition), and so* $s_i = \widetilde{s} + d$ *is the greatest element satisfying* $\delta(s_i) = 0$.
(For this reason $\widetilde{s} + d$ is a "good candidate" for s_m).

2. *If* $s_i \geq 2d'$ *we know that* $\gamma(s_i) \leq 0$ *and easily one can see when* $\nu(s_{i+1}) < \nu(s_i)$.

3. *If* $s_i < 2d'$ *we can write* $\nu(s_{i+1}) - \nu(s_i)$ *in function of* $\gamma(s_i)$*: it depends also on the facts:*

   $s_{i+1} - c \in S$ *or* $\notin S$,

   $s_i - d \quad \in S$ *or* $\notin S$,

   $s_{i+1} - c' \in S$ *or* $\notin S$.

In [22] the results on the position of s_m are explained by means of several tables. For example we show for $s_i < 2d'$ how the difference $\eta(s_i) := \nu(s_{i+1}) - \nu(s_i)$ depends on the value of $\gamma := \gamma(s_i)$. In the following table $\times$ means $\in S$ and $\bigcirc$ means $\notin S$. Assume $s_i \leq 2d' - 1$ Then:

$s_{i+1} - c$	$s_i - d$	$s_{i+1} - c'$	α	β	δ	$\eta(s_i)$
$\bigcirc$	$\times$	$\bigcirc$	-2	0	0	$\gamma - 2$
$\bigcirc$	$\times$	$\times$	0	0	0	γ
$\bigcirc$	$\bigcirc$	$\bigcirc$	0	0	0	γ
$\times$	$\times$	$\bigcirc$	-2	0	2	γ
$\bigcirc$	$\bigcirc$	$\times$	2	0	0	$\gamma + 2$
$\times$	$\bigcirc$	$\bigcirc$	0	0	2	$\gamma + 2$
$\times$	$\times$	$\times$	0	0	2	$\gamma + 2$
$\times$	$\bigcirc$	$\times$	2	0	2	$\gamma + 4$

Recall: $\gamma(s_i)$ concerns pairs $(x, y) \in \mathbb{N}(s_i) \cap [0, d']^2$.

2.2 Evaluation or bounds for s_m.

Theorem 2.9 $(See\ [22])$ *With setting (2.3) we have:*

1. *If* $\widetilde{s} < 2d' - d$, *then* $s_m \leq 2d'$.

   If, moreover, $[\widetilde{s} + 2, d'] \cap \mathbb{N} \subseteq S$, *then* $s_m = \widetilde{s} + d$.

2. *If* $\widetilde{s} \geq 2d' - d$, *then* $s_m \leq \widetilde{s} + d$.

   More precisely:

   (a) *If* $\widetilde{s} \geq d' + c' - d$, *then* $s_m = \widetilde{s} + d$.

   (b) *If* $\widetilde{s} = 2d' - d$, *then* $s_m = \widetilde{s} + d$.

   (c) *If* $2d' - d < \widetilde{s} < d' + c' - d$, *we can give upper and lower bounds for* s_m *under additional assumptions. In particular:*

   if $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$, *then* $c + d - e \leq \widetilde{s} + d' - \ell + 1 \leq s_m \leq 2d'$.

   Case (a) is satisfied e.g. if $d - 2 \leq \widetilde{s} \leq d$; *or if* $c' \leq \widetilde{s} \leq d$, *in particular if* S *is acute.*

Example 2.10

1. $S = \{0, 25_e, 26, 28, 30, 31_{d'}, 33_d, 39_c \rightarrow\}$
   $(\widetilde{s} = 28,\ \widetilde{s} < 2d' - d,\ [\widetilde{s} + 2, d'] \cap \mathbb{N} \subseteq S,\ s_m = \widetilde{s} + d)$.

2. $S = \{0, 7_{e=d'}, 13_{c'}, 14, 15, 16, 17_d, 20_c \rightarrow\}$
   (S is acute, $\ell = 2$, $\widetilde{s} = 14$, $c' \leq \widetilde{s} \leq d$, $\widetilde{s} > d' + c' - d$).

3. $S = \{0, 20_e, 21, 26, 27_{d'}, 32_d, 39_c \rightarrow\}$
   $(\widetilde{s} = 21 < 2d' - d,\ s_m = 2d' = 54 > \widetilde{s} + d)$.

4. $S = \{0, 10_e, 20, 22, 23_{d'}, 26_d, 30_c \rightarrow\}$
   $(2d' - d < \widetilde{s} = 22 < d' + c' - d,\ \ s_m = 46 < \widetilde{s} + d)$.

2.3 Conjecture and particular cases.

We believe the following fact is true for each semigroup.

Conjecture: $s_m \geq c + d - e \qquad (*)$

We proved in [22] that $(*)$ holds in several cases, in particular

1. *If either* $(s_m \geq \widetilde{s} + d)$ *or* $(s_m \geq 2d'$ *and* $\widetilde{s} < d')$.
2. *If* $2d' - d < \widetilde{s} < d' + c' - d$ *and* $[d' - \ell, d'] \cap \mathbb{N} \subseteq S$ *(2.9.2c)*.
3. *When* $\ell = 2$, *or* $\ell = 3$ *(here we calculate* s_m *exactly).*
4. *If* $\tau \leq 7$
   $($*where* $\tau := \#\{x \in \mathbb{N} \setminus S \mid x + (S \setminus \{0\}) \subseteq S\}$ *is the Cohen-Macaulay type of* $S)$.
5. *If* $e \leq 8$ *(by* (4)*, since* $\tau \leq e - 1)$.
6. *If* S *is generated by a* **generalized arithmetic sequence** $($*i.e.* $S =< m_0, m_1, ..., m_p >$ *where* $m_i = am_0 + id$, *for some* $a \geq 1, d \geq 1)$, *then* $s_m = \widetilde{s} + d$ *and so* $(*)$ *holds.*
7. *If* S *is generated by an* **almost arithmetic sequence** $($*i.e.* $S =< m_0, m_1, ..., m_k,\ n >$*, where* $m_0, m_1, ..., m_k$ *is an arithmetic sequence*$)$ *and embdim*$(S) \leq 5$,
   then $s_m \geq c + d - e$.

3 Weierstrass Semigroups.

In this section we deal with the following

Question : Which numerical semigroups are Weierstrass?

The problem to find conditions in order that a semigroup is Weierstrass seems to be very hard: there are only partial answers in several directions. Most of them are in $characteristic\ 0$, so we fix the following

Setting 3.1 *From now on we assume that* $\mathbb{F}$ *is algebraically closed with* $char(\mathbb{F}) = 0$.

We know that there exist non-Weierstrass semigroups: the first example is due to an idea of Buchweitz:

Example 3.2 (See [2]) Let $S =< 13, 14, 15, 16, 17, 18, 20, 22, 23 >$, with $g = 16$, $c = 26$, $H = \mathbb{N} \setminus S = \{1, ..., 12, 19, 21, 24, 25\}$.

S cannot be Weierstrass. In fact assume that there exist a curve X and a point $P \in X$ such that $S = S(P)$. Then, by Remark 1.2, X would have regular differentials ω_i vanishing at P to orders $i = \text{ord}_P(\omega_i)$ with $i \in \{0, 1, 2, \ldots, 10, 11, 18, 20, 23, 24\}$.
Hence, taking suitable (tensor) products of the differential forms above, X would have also at least 46 linearly independent "quadratic" differentials vanishing to every order $\in \{0, ..., 35, 36, 38, 40, 41, 42, 43, 44, 46, 47, 48\}$ at P. This implies that $\lambda(2K) \geq 46$, a contradiction since, by Riemann-Roch it is $\lambda(2K) = 3g - 3 = 45$.

There are generalizations of this idea due to Kim [15] and Komeda [16]:

Proposition 3.3 [16] *For a semigroup S of genus g, let $\mathbb{N}\setminus S = \{h_1, \dots, h_g\}$ and let*

$$H_m := \{h_{i_1} + \cdots + h_{i_m} \mid 1 \le i_j \le g\},\ m \ge 2.$$

If S is Weierstrass, then $\#\, H_m \le (2m-1)(g-1)$ $for\ each\ m \ge 2$ $\qquad (**)$

Proof. If S is the Weierstrass semigroup of a curve X at P, then X has regular differentials vanishing to order $h_i - 1$, $\forall i = 1, .., g$. In fact let K be a canonical divisor (in particular deg$(K) = 2g-2$): for each $h_i \in \mathbb{N} \setminus S,\ \lambda(h_i P) = \lambda((h_i - 1)P)$ therefore by Riemann-Roch

$$\lambda(K - (h_i - 1)P) > 0.$$

It follows $\lambda(mK) \ge \#H_m,\ \forall m \ge 2$, because $\forall y_j \in H_m$, $\mathcal{L}(mK)$ contains a m-differential vanishing to order $(y_j - m)$ at P. Now it suffices to recall that, again by Riemann-Roch, $\lambda(mK) = (2m-1)(g-1)$. $\diamond$

Remark 3.4 *The conditions $(**)$ of (3.3) are satisfied for each $m \ge 2$, if $2c < 3g$.*

Proof. Since $\mathbb{N} \setminus S \subseteq [1, c-1] \cap \mathbb{N}$ we get $\#(H_m) \le m(c-1)$, then the inequality $(**)$ in (3.3) is surely satisfied if $m(c-1) \le (2m-1)(g-1)$ for each $m \ge 2$. This condition is equivalent to $mc \le (2m-1)g - (m-1)$: for $m = 2$, get $2c \le 3g - 1$, i.e. $c \le 3g/2 - 1/2$. Now assume $2c \le 3g - 1$, and so $mc \le 3mg/2 - m/2\ \forall m > 0$: one can easily see that the inequality $3mg/2 - m/2 \le (2m-1)g - (m-1)$ holds $\forall m \ge 2,\ \forall g > 0$. $\diamond$

Remark 3.5 *In Buchweitz's example, $m = 2$, $g = 16$, $\#H_2 = 46 > 3g - 3$. Note that for $m = 2$, the genus $g = 16$ is the "minimum" example: in fact Komeda and Tsuyumine found by a direct computation that*

$$\textit{for each } 2 \le g \le 15 \textit{ we have } \#H_2 \le 3g - 3.$$

F.Torres found a method to construct symmetric non-Weierstrass semigroups (of large genus):

Example 3.6 $(See$ [33]$)$ Let S' be a non-Weierstrass semigroup of genus γ, and let $g \in \mathbb{N},\ g \ge 6\gamma + 4.$ Then the following symmetric semigroup:

$$S = \{2s \mid s \in S'\} \cup \{2g - 1 - 2t \mid t \in \mathbb{Z} \setminus S'\}$$

is *non-Weierstrass.*

This fact is true since we have:

Proposition 3.7 [33, Scholium 3.5] *Assume that a semigroup S of genus $g \ge 6\,\gamma + 4$, is γ-hyperelliptic, i.e. satisfies*

1. *the first γ elements $m_1, ..., m_\gamma \in S,\ m_i > 0,$ are even;*

2. $m_\gamma = 4\gamma$*;*

3. $4\gamma + 2 \in S$.

Then: S Weierstrass $\Longrightarrow S' := \left\{0, \dfrac{m_1}{2}, ..., \dfrac{m_\gamma}{2}\right\} \cup_{i \in \mathbb{N}} \{2\gamma + i\}$ is Weierstrass.

Example 3.8 [21] The possible Weierstrass semigroups for a plane smooth projective quintic (hence of genus 6) are of the following types:

$$S_1 = < 4, 5 >$$

$$S_2 = < 4, 7, 10, 13 >$$

$$S_k = \{0, 6 \rightarrow\} \setminus \{k\} \; with \; 6 \leq k \leq 11.$$

Note: $S_1,\ S_2$ and S_k for $k = 6, 11$ are semigroups generated by an arithmetic sequence, and S_k is generated by an almost arithmetic sequence for $k = 10$.

3.1 Deformations and $T^1(\mathcal{O}_{X,O})$.

The next theorem due to Pinkham (thesis) is fundamental to approach our question.

Theorem 3.9 [25] *Let S be a numerical semigroup and let $X = Spec(\mathbb{F}[S])$ be the monomial curve associated to S. Then:*

S is Weierstrass if and only if X is smoothable

We want to recall the main tools of the theory. Recall that the field $\mathbb{F}$ is algebrically closed with char$(\mathbb{F}) = 0$. We collect here the most important results and definitions on deformations of algebraic varieties.

Definition 3.10 *A deformation $\pi : Y \longrightarrow \Sigma$ of a variety X is a cartesian diagram*

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \big\downarrow & & \big\downarrow \pi \\ \{0\} & \hookrightarrow & \Sigma \end{array}$$

where π is a flat morphism.
A deformation $\pi : Y \longrightarrow \Sigma$ of X is said to be versal if any deformation $\pi' : Y' \longrightarrow \Sigma'$ of X is isomorphic to a deformation obtained from π by a base change $h : \Sigma' \longrightarrow \Sigma$:

$$\begin{array}{ccc} Y' = Y \times \Sigma' & \xrightarrow{pr_1} & Y \\ \pi' \big\downarrow & & \big\downarrow \pi \\ \Sigma' & \xrightarrow[h]{} & \Sigma \end{array}$$

When $\Sigma = Spec\ k[\varepsilon]/(\varepsilon^2)$ we say that the deformation is infinitesimal.
Finally we say that a deformation is trivial if $Y \simeq X \times \Sigma$.

Definition 3.11 *A variety X is smoothable if there exists a deformation Y of X having smooth generic fibre.*

For a survey on deformations we refer to [31]. We recall the main theorem

Theorem 3.12 [25]
If X is affine variety and has an isolated singularity, then there exists a versal deformation Y of X. Further, if X has a $\mathbb{G}_m$-action, then there exists a $\mathbb{G}_m$-action on Y extending the action on X.

Corollary 3.13 *Let $X = Spec(\mathbb{F}[S])$, S a numerical semigroup. Then X has a versal deformation Y compatible with the well-known $\mathbb{G}_m$-action.*

Notation 3.14 *Let $S =< n_0, ..., n_k >$ be a semigroup, $P = \mathbb{F}[x_0, ..., x_k]$, weight$(x_i) := n_i \quad (0 \leq i \leq k)$, and let*

$$B := \mathbb{F}[S] = \mathbb{F}[t^{n_0}, t^{n_1}, ..., t^{n_k}] = P/I,$$

where $\ I = (f_1, ..., f_q), \ \ f_i$ homogeneous binomials, $\ d_i :=$ deg$(f_i), \ \forall i = 1, ..., q$. Further let:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} \in P^q$$

$$G_\ell := \{i \in \{0, \dots, k\} \mid n_i + \ell \notin S\}$$

$$H_\ell := \{d_k, \ k \in \{1, \dots, q\} \mid d_k + \ell \notin S\} \ (\ell \in \mathbb{Z}).$$

In order to construct deformations for the curve X we need the B-$module$ T^1_B. Let $\Omega_{P/\mathbb{F}}$ be the P-$module$ of 1-$differential\ forms$, then $\ Hom_B(\Omega_{P/\mathbb{F}} \otimes B, B)\ $ is a free B-$module$ generated by the partial derivatives $\ < \frac{\partial}{\partial x_0}, ..., \frac{\partial}{\partial x_k} >$:

Definition 3.15 *Consider the map*

$$\begin{array}{rccc} \varphi : & Hom_B(\Omega_{P/\mathbb{F}} \otimes B, B) & \longrightarrow & Hom_B(I/I^2, \ B) \\ & \frac{\partial}{\partial x_i} & \mapsto & g \ : \ g(f) = (\frac{\partial f}{\partial x_i}) \quad (mod\ I) \end{array}$$

We define the B-module T^1_B as $\ T^1_B := Coker\ \varphi$.

Let f be as in (3.14): we shall identify a map $g \in Hom_B(I/I^2, \ B)$ with the column vector $(h_j)_{j=1,...q} := (g(f))$ of its image mod I.

Remark-Notation 3.16 *Let J_0 be the jacobian matrix of deg 0-$derivatives$: $J_0 = \left(\ x_i \dfrac{\partial f_j}{\partial x_i} \ \right)$. Then*

$$J_0 \equiv \begin{pmatrix} t^{d_1} & 0 & \dots & 0 \\ 0 & t^{d_2} & \dots & 0 \\ & \dots & \dots & \\ & \dots & \dots & \\ 0 & 0 & \dots & t^{d_q} \end{pmatrix} J_0(1) \quad (mod\ I).$$

where $J_0(1)$ is the evaluation of J_0 at the regular point $Q(1, \dots, 1) \in X$: by the jacobian criterion of regularity we know that $\ \ rank\ J_0(1) = k$.
Further for $\ell \in \mathbb{Z}$, let $\ J_\ell\ $ denote the submatrix of $\ J_0(1)\ $ obtained by considering the rows $\left(\dfrac{\partial f_i}{\partial x_j}(1, ..., 1)\right)$ with $\ d_i \in H_\ell$.

Proposition 3.17 *1.* $T^1_B = \bigoplus_{\ell\in\mathbb{Z}} T^1_B(\ell)$ *is a* $\mathbb{Z}$-$graded$ $\mathbb{F}$-$vector\ space$ *of finite dimension:*

$\overline{g} \in T^1_B(\ell) \Longleftrightarrow$ *deg*$(g(f_j)) =$ *deg*$(f_j) + \ell\ \ \forall j$ *(see* [25]*).*

2. *For* $g \in T^1_B :\ \ g = \sum_{i=1}^k \alpha_i t^{\beta_i} \dfrac{\partial}{\partial x_i}$.

3. *dim*$_{\mathbb{F}} T^1_B(\ell) = \#G_\ell -$ *dim* $V_\ell - 1$, *where* V_ℓ *is the sub-vector-space of* $\mathbb{F}^{k+1}$ *generated by the row-vectors of the matrix* J_ℓ.

Proof. 2. We know that there exists $n \in \mathbb{N}$ such that $m^n T^1_B = 0$. Therefore for each $g \in Hom_B(I/I^2, B)$ there exists $a \in \mathbb{N}$ such that $t^a g \in Im\phi$. Further in B the Euler's identity holds: $\sum_{i=0}^k n_i\, x_i \dfrac{\partial f_j}{\partial x_i} = 0,\ \forall j = 1, \ldots, q$ and so g can be rewritten as a linear combination of the partial derivatives with respect to $(1, \ldots, k)$.
3. Recall that $Im(\varphi)$ is generated by the the partial derivatives. For each $i \notin G_\ell$, we have: $t^{\ell+n_i} \in B$ and so $t^{\ell+n_i} \dfrac{\partial}{\partial x_i} \in Im\ \Phi$. On the contrary, note that for $i \in G_\ell$, the vector

$$\sum_1^k \alpha_i t^{\ell+n_i} \frac{\partial}{\partial x_i} \in Hom_B(I/I^2, B)(\ell) \Longleftrightarrow J_\ell\,(0, \alpha_1, \alpha_2, \ldots, \alpha_k)^T = 0,\ \ \forall\ d_j \in H_\ell.$$

Therefore the system has $\infty^{\#G_l - \text{dim } V_\ell - 1}$ solutions. $\diamond$

For a semigroup S, let $S(1) := \{n \in \mathbb{Z} \mid n + n_i \in S\ \ \forall i \geq 0\}$.

Proposition 3.18 *Let* $L = S(1) \cup \{n \in \mathbb{Z} \mid n < -2c + 2 - 2n_0\}$. *Then*

1. *dim* $T^1_B(\ell) = 0$ *for each* $\ell \in L$.
2. *in particular if* S *is ordinary or hyperelliptic, then dim* $T^1_B(\ell) = 0$ *for each* $\ell < -4g - 2$.
3. *Let* $f'_1, \ldots, f'_q$ *be a reordering of the set* $\{f_1, \ldots, f_q\}$ *such that the degrees satisfy* $d'_1 \leq d'_2 \leq \cdots \leq d'_q$. *Let* $J'_0(1)$ *be the associated jacobian matrix and let* $p = minimum$ *integer such that the first* p *rows of* $J'_0(1)$ *constitue a matrix of rank* $= k$. *Then* $T^1_B(\ell) = 0$ *for each* $\ell < -d'_p$.
4. *dim* $T^1_B(c - 1 - n_0 - n_1) > 0$ $(see$ [25]$)$.
5. *If* $\ell \geq c - 2n_1$, *then dim* $T^1_B(\ell) =$ *max*$\{0, \#G_\ell - 1\}$
6. *If* $\ell \geq c - 1 - n_0$, *then dim* $T^1_B(\ell) = 0$.
7. *For each* $i = 0, \ldots, k$ *we have:* $t^{c+n-n_i} \dfrac{\partial}{\partial x_i} \in Im\ \Phi,\ \ \forall n \geq 0$.

Proof. 1. If $\ell \in S(1)$, then $G_\ell = \emptyset$ and we are done by (3.17.3)
If $\ell < -2c+2-2n_0$, then $\#G_\ell = n_0$ and $n_i+n_j+\ell \leq 2(n_0+c-1)+\ell < 0$, hence $H_\ell = \{1, \ldots, q\}$. Then dim $V_\ell = n_0 - 1$ and the claim follows by (3.17.3).
2. Follows by (1): if S is ordinary, or hyperelliptic, then $-2c - 2n_0 = -4g - 2$.

3. Immediate by the assumptions and by (3.17.3), since we have: dim $V'_\ell = n_0 - 1$, because $H_\ell \supseteq \{1, \dots, p\}$.
4. Let $\ell = c - 1 - n_0 - n_1$: we have $\{0, 1\} \subseteq G_\ell,$ while $H_\ell = \emptyset$ since $d_i > n_0 + n_1 \;\; \forall i$. Therefore the claim follows by (3.17.3).
5. Follows by (3.17.3) since in this case $H_\ell = \emptyset$.
6. Follows by (3.17.3) as a particular case.
7. Recall that $t^{c+n} \in B$ and deg $\left(\dfrac{\partial}{\partial x_i}\right) = -n_i$. $\diamond$

For flatness conditions an essential fact is the following:

Proposition 3.19 (*See, e.g.* [31, Page 8]) *Given a cartesian diagram*

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \\ \{0\} & \hookrightarrow & \Sigma \end{array}$$

let f_i and $F_i,\ i = 1, \dots, q$ be respectively the equations of X and Y. Then:
the map π is flat $\iff$ every relation $\sum_1^q r_i f_i = 0,\ \ r_i, f_j \in \mathbb{F}[x_0, \dots, x_k]$ can be lifted to a relation $\sum_1^q R_i F_i = 0,\ \ R_i, F_j \in \mathbb{F}[x_0, \dots, x_k] \otimes \mathcal{O}_{\Sigma,0}$.

Theorem 3.20 (See, e.g., [31]) *The infinitesimal deformations are in one-to-one correspondence with* $Hom_B(I/I^2,\ B)$ *as follows*

$$\begin{pmatrix} g : I/I^2 \longrightarrow B \\ f_j \mapsto g_j(\textit{mod}\ I) \\ (j = 1, ..., q) \end{pmatrix} \begin{array}{c} corresponds \\ to\ the \\ deformation \end{array} \quad F = \begin{pmatrix} f_1 + \varepsilon g_1 \\ \dots \\ f_q + \varepsilon g_q \end{pmatrix}$$

Proof. (Outline) The trivial deformations (i.e. $Y \simeq \Sigma \times X$) lie in $Im\ \varphi$ (3.15).

In fact these deformations are such that the ideal generated by the $(f_i + \varepsilon g_i) \in \mathbb{F}[\varepsilon, x_0, \dots, x_k]$ becomes equal, after a change of variables, to the ideal generated by the (f_i). Now note that a change of variables is $\{x_i \mapsto x_i + \varepsilon h_i\}$; since $\varepsilon^2 = 0$ easily one can see that it adds to each g_i an element of the form $\sum_{j=0}^k \dfrac{\partial f_i}{\partial x_j} h_j$. Therefore T_B^1 can be naturally identified with the set of infinitesimal deformations modulo the trivial ones. $\diamond$

Several semigroups have been recognized to be Weierstrass by means of the above theory: we collect in the following theorem the most important statements.

Theorem 3.21 *Assume X be an affine curve.*

1. *If X is a complete intersection then X is smoothable* [27].

2. *If $X \subseteq \mathbb{A}^3$ or X is a Gorenstein curve of embedding dimension 4, then X is smoothable* [*29*], [*3*].

3. *Let $e,\ g$ denote respectively the multiplicity and the genus of the semigroup S and let $X =$ Spec$(\mathbb{F}[S])$. Then:*

(a) If $e \in \{3,4,5\}$*, then* S *is Weierstrass: for* $e=3$*, see also* [14]*, for* $e=4$*,* $e=5$*, see* ([16], [17])*.*

(b) If $g \leq 8$*, then S is Weierstrass ([18]).*

(c) If $2e > c-1$ *and* $g=9$*, then S is Weierstrass ([19]).*

4. *Let* $H = \mathbb{N} \setminus S$*, define* $weight(S) := \sum_{i=1}^{g} h_i - i$*: if* $weight(S) \leq g/2$*, then* S *is Weierstrass* ([7])*.*

5. *If* B *is negatively graded* (*i.e.* $T^1_B(\ell) = 0$ *for each* $\ell \geq 0$)*, then* S *is Weierstrass* ([26])*.*

3.2 Construction of the versal deformation with $\mathbb{G}_m$-action.

With the above notations for the monomial curve $X := \mathrm{Spec}(\mathbb{F}[S])$, S a numerical semigroup, we shall describe Pinkham's algorithm [25] to construct a deformation Y admitting a $\mathbb{G}_m$-action. Starting from the infinitesimal deformation associated to $\bigoplus_{\ell<0} T^1_B(\ell)$, by means of a finite number of steps one can obtain such deformation (with the greatest parameter space). Each step consists in the lifting of a deformation on $\Sigma = \mathrm{Spec}\ \mathbb{F}[\varepsilon]/(\varepsilon)^n$ to a deformation on $\Sigma' = \mathrm{Spec}\ \mathbb{F}[\varepsilon]/(\varepsilon)^{n+1}$. Further in the last step we recall Pinkham's construction (when possible) of a projective regular curve $\mathcal{C}$ admitting S as semigroup at the point P_∞ (see [25, 13.3]). This construction is the main ingredient for the proof of Theorem 3.9.

Step (0) The first step of the algorithm is the explicit computation of a $\mathbb{F}$-basis E for T^1_B.

Step (1) Let r be a $(p \times q)$ matrix of relations among the generators $\{f_i\}$ of I.
For each $g_j \in E$ construct a $(p \times q)$ matrix $\rho_j = \rho_j(x_0, ..., x_k)$, such that $R = r + \varepsilon\rho_j$ is a relation matrix among the equations of $F = f + \varepsilon g_j$, i.e.,

$$(r + \varepsilon\rho_j)(f + \varepsilon g_j) = rf + \varepsilon(rg_j + \rho_j f) = 0 \pmod{\varepsilon^2}.$$

A matrix ρ_j such that $\rho_j f = -rg_j$ exists since any $g \in Hom_B(I/I^2, B)$ is a derivation (3.15.2), and so the matrix rg has entries $\in I$, for each $g \in Hom_B(I/I^2, B)$. In fact if $\sum r_i f_i = 0$, then $0 = g(\sum r_i f_i) = \sum r_i g(f_i) + \sum g(r_i) f_i = \sum r_i g(f_i) \bmod I$, *i.e.* $\sum r_i g(f_i) \in I$.
Hence any relation among the (f_i) lifts to a relation R among the (F_i), so that the projection π is flat (3.19). Let $E =< g_1, ..., g_m >$ be the $\mathbb{F}$-basis of $\bigoplus_{\ell<0} T^1_B(\ell)$: assign a parameter U_j to each g_j with

$$\mathrm{weight}(U_j) := -\mathrm{deg}(g_j).$$

We obtain homogeneous equations

$$F = f + \varepsilon(g_1 U_1 + ... + g_m U_m) \in \mathbb{F}[U_1, ..., U_m, x_0, ..., x_k]$$

for a deformation Y_1 of X with base space Spec $\mathbb{F}[\varepsilon]/(\varepsilon)^2$.
By linearity the matrix $\rho := U_1\rho_1 + ... + U_m\rho_m$ is such that $r + \varepsilon\rho$ is a relation matrix for F.

Step (2) Now, called $g := (g_1 U_1 + ... + g_m U_m)$, look for a vector h and for a matrix ρ' such that

$$F = f + \varepsilon g + \varepsilon^2 h \quad \text{and} \quad R = r + \varepsilon\rho + \varepsilon^2\rho'$$

verify

$$RF = rf + \varepsilon(rg + \rho f) + \varepsilon^2(\rho g + rh + \rho' f) \equiv 0 \mod (\varepsilon)^3$$
$$RF = \varepsilon^2(\rho g + rh + \rho' f) \equiv 0 \mod (\varepsilon)^3$$

Note that ρg is quadratic in $U_1, ..., U_m$, therefore both ρ', h will be quadratic in $U_1, ..., U_m$. To solve this equation we must impose several conditions to the variables $\{U_1, ..., U_m\}$, but a solution exists since X has a versal deformation by (3.12).

....

Step (n) The matrices to find have entries of $degree\ n$ in $U_1, ..., U_m$. We already know that the algorithm ends. Surely it ends when deg$(U_{i_1}...U_{i_h}) >$ deg(f_j) $\forall\ j$ and $\forall\ (i_1, ..., i_h)$. In fact at this step the needed matrices are null by the theorem of existence of a versal deformation for X admitting a $\mathbb{G}_m$-action [25].

Last Step Let $R := \mathbb{F}[U_1, ..., U_n]/J$, $\Sigma =$ Spec(R) be the parameter space of the constructed deformation Y of X with $\mathbb{G}_m$-*action* and let $F = f + U_1 g_1 + ... + U_m g_m + U_1^2 h_{11} + \dots$ be the defining equations of Y. Substitute $U_i x_{k+1}^{\text{weight}(U_i)}$ for U_i and let $A := R[x_0, \dots, x_{k+1}]/(F)$. Then the morphism $\pi : Proj(A) \longrightarrow \Sigma$ is flat and proper with fibres reduced projective curves [25, 13.4]. The generic fibre $\mathcal{C}$, has only one regular point $P_\infty(t^{n_0} : t^{n_1} : \dots : t^{n_k} : 0)$ at infinity. If one fibre $\mathcal{C}$ is regular, then the semigroup associated to the pair $(\mathcal{C}, \mathcal{P}_\infty)$ is clearly equal to the semigroup S.

4 Examples

In this section we show the above algorithm in some particular example.

4.1 The case of embedding dimension 3

First we calculate explicitely a deformation with $\mathbb{G}_m$-action for a monomial curve $X \subseteq \mathbb{A}^3_{\mathbb{F}}$.

Example 4.1 *Let $S =< 4, 9, 11 >$, $B = \mathbb{F}[S]$, $X :=$ Spec B.*
The conductor is $c = 15$, the Apery set is $\mathcal{A} = \{n_0 = 4, n_1 = 9, n_2 = 11, n_3 = 18\}$.
The equations defining the curve X in $\mathbb{F}[x_0, x_1, x_2]$ are

$$f_1 = x_0^5 - x_1 x_2, \quad f_2 = x_0 x_1^2 - x_2^2, \quad f_3 = -x_1^3 + x_0^4 x_2$$

with matrix of relations: $r = \begin{pmatrix} -x_2 & x_1 & x_0 \\ x_1^2 & -x_0^4 & -x_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ *and Jacobian matrix*

$$J_0 = \left(x_0 \frac{\partial f_j}{\partial x_0}, x_1 \frac{\partial f_j}{\partial x_1}, x_2 \frac{\partial f_j}{\partial x_2} \right) = \begin{pmatrix} 5x_0^5 & -x_1 x_2 & -x_1 x_2 \\ x_0 x_1^2 & 2x_0 x_1^2 & -2x_2^2 \\ 4x_0^4 x_2 & -3x_1^3 & x_0^4 x_2 \end{pmatrix}$$

$$J_0 \equiv \begin{pmatrix} t^{20} & 0 & 0 \\ 0 & t^{22} & 0 \\ 0 & 0 & t^{27} \end{pmatrix} \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & -2 \\ 4 & -3 & 1 \end{pmatrix} \qquad (mod\ I).$$

Let $\Delta_i := x_i \frac{\partial}{\partial x_i},\ i = 0,1,2$ (degree 0 derivations).

Step (0) One can easily see that $T^1(B)$ is generated as B-module by

$$\begin{array}{ll} T^1(-18): & t^{-18}(\Delta_1 - \Delta_2) := D_1 \\ T^1(-16): & t^{-16}(\Delta_1 + \Delta_2) := D_2 \\ T^1(-11): & t^{-11}(\Delta_1 + \Delta_2) := D_3. \end{array}$$

with images the classes mod I of

$$< g_1 = \begin{pmatrix} 0 \\ 4x_0 \\ -4x_1 \end{pmatrix},\ g_2 = \begin{pmatrix} -2x_0 \\ 0 \\ -2x_2 \end{pmatrix},\ g_3 = \begin{pmatrix} -2x_1 \\ 0 \\ -2x_0^4 \end{pmatrix} >$$

(Note that as $\mathbb{F}$-vector spaces we have $\dim_{\mathbb{F}} T^1(B) = 17,\ \ \dim_{\mathbb{F}} T^1(B)^- = 15$).

Step (1) Using the above algorithm (restricted to three generators) we get the infinitesimal deformation

$$\pi : \mathrm{Spec}\big(\mathbb{F}[\varepsilon]/(\varepsilon)^2 \otimes \mathbb{F}[x_0,x_1,x_2]/I_1\big) \longrightarrow \mathrm{Spec}\big(\mathbb{F}[\varepsilon]/(\varepsilon)^2\big),$$

with $U_i \in \mathbb{F},\ I_1$ generated by the rows of $F_1 = f + \varepsilon g$, with $g = U_1 g_1 + U_2 g_2 + U_3 g_3$:

$$F_1 = f + \varepsilon \left[U_1 \begin{pmatrix} 0 \\ x_0 \\ -x_1 \end{pmatrix} + U_2 \begin{pmatrix} x_0 \\ 0 \\ x_2 \end{pmatrix} + U_3 \begin{pmatrix} x_1 \\ 0 \\ x_0^4 \end{pmatrix} \right],\ \text{weight}(U_1,U_2,U_3) = (18,16,11).$$

In fact there exists the matrix $\rho = \begin{pmatrix} -U_3 & 0 & 0 \\ U_1 & -U_2 & U_3 \end{pmatrix}$ such that $(r+\varepsilon\rho)(f+\varepsilon g) \equiv 0$ (mod $(\varepsilon)^2$, i.e., $(rg + \rho f = 0)$ (this assures π is flat, with $R_1 := r + \varepsilon\rho$ relation matrix for F_1):

$$rg = \begin{pmatrix} U_3(x_0^5 - x_1x_2) \\ U_1(-x_0^5 + x_1x_2) + U_2(x_0x_1^2 - x_2^2) + U_3(-x_1^3 + x_0^4x_2) \end{pmatrix} = -\rho f.$$

Step(2) Now look for h, ρ' such that $F_2 = f + \varepsilon g + \varepsilon^2 h$ and $R_2 = r + \varepsilon\rho + \varepsilon^2\rho'$ satisfy $F_2R_2 = 0$ (mod $(\varepsilon)^3$), i.e., $\rho g + rh + \rho' f \equiv 0$. Get

$$\rho g = \begin{pmatrix} -U_2U_3x_0 - U_3^2x_1 \\ -U_1U_2x_0 - U_1U_3x_1 + U_1U_2x_0 + U_2U_3x_2 + U_1U_3x_1 + U_3^2x_0^4 \end{pmatrix} =$$

$$= \begin{pmatrix} -x_2 & x_1 & x_0 \\ x_1^2 & -x_0^4 & -x_2 \end{pmatrix} \begin{pmatrix} 0 \\ -U_3^2 \\ -U_2U_3 \end{pmatrix} = -rh, \quad \text{with} \quad h = \begin{pmatrix} 0 \\ U_3^2 \\ U_2U_3 \end{pmatrix}.$$

Finally one can see that $\rho h = 0$, therefore we can choose $\rho' = 0$. Hence the algorithm ends at the second step and a deformation of f on Spec $\mathbb{F}[U_1,U_2,U_3]$ has homogeneous weighted equations

$$F = f + \begin{pmatrix} U_2x_0 + U_3x_1 \\ U_1x_0 \\ -U_1x_1 + U_2x_2 + U_3x_0^4 \end{pmatrix} + \begin{pmatrix} 0 \\ U_3^2 \\ U_2U_3 \end{pmatrix}.$$

Remark 4.2 Note that in the entries of the matrix h the coefficient of U_1^2 is null. This is clear since $\deg(U_1^2) = 36 > \deg(f_i),\ \forall\ i = 1, 2, 3,$ and the equations are homogeneous according to the existence of a $\mathbb{G}_m$-action. Hence if we restrict to g_1, we get the deformation

$$\pi : Y = \operatorname{Spec}\big(\mathbb{F}[U_1] \otimes \mathbb{F}[x_0, x_1, x_2]/J\big) \longrightarrow \mathbb{A}^1_{\mathbb{F}}$$

with the ideal J generated by the rows of

$$F_1 = \begin{pmatrix} x_0^5 - x_1x_2 \\ x_0x_1^2 - x_2^2 \\ -x_1^3 + x_0^4x_2 \end{pmatrix} + U_1 \begin{pmatrix} 0 \\ x_0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} x_0^5 - x_1x_2 \\ x_0x_1^2 - x_2^2 + U_1x_0 \\ -x_1^3 + x_0^4x_2 - U_1x_1 \end{pmatrix}.$$

The algorithm ends at step (1) with smooth parameter space $\mathbb{A}^1_{\mathbb{F}}$. The Jacobian matrix of the generic fiber of π is

$$\begin{bmatrix} 5x_0^4 & -x_2 & -x_1 \\ x_1^2 + U_1 & 2x_0x_1 & -2x_2 \\ 4x_0^3x_2 & -3x_1^2 - U_1 & x_0^4 \end{bmatrix}$$

One can check that the generic fiber is non singular.

The general 3-Space case. By means of a costruction due to Patil-Singh [23] we can compute directly the equations of the monomial curve associated to a semigroup $S =< n_0, n_1, n_2 >$. In this case we already know that every semigroup singularity is smoothable by Shaps' paper [29]: here the equations of a deformation are obtained as minors of a suitable matrix. Let $S =< n_0, n_1, n_2 >$, $n_0 < n_1 < n_2$, let $Ap(S)$ be the Apery set respect to n_0 and let

$$\begin{cases} u := \min\{n \in \mathbb{N} \mid un_1 \in < n_0, n_2 >,\ \ un_1 \notin Ap(S)\} \\ v := \min\{n \in \mathbb{N} \mid vn_2 \in < n_0, n_1 >\} \end{cases}$$

$$\text{Then} \begin{cases} un_1 = \lambda n_0 + wn_2,\ \lambda \geq 1 \\ vn_2 = \mu n_0 + zn_1,\ \ v \geq 2,\ \ v > w,\ 0 \leq z < u \\ \quad \text{further:} \\ (\lambda + \mu)n_0 = (u - z)n_1 + (v - w)n_2. \end{cases} \qquad (*).$$

By [23] we know that the curve is a complete intersection $\iff zw\mu = 0$.
Then assume $zw\mu \neq 0$: we get the following generators for the ideal I and the relation module r of X:

$$I = \begin{cases} f_1 = x_1^u - x_0^\lambda x_2^w \\ f_2 = x_1^{u-z}x_2^{v-w} - x_0^{\lambda+\mu} \\ f_3 = x_2^v - x_0^\mu x_1^z \end{cases} ; \quad r = \begin{pmatrix} -x_2^{v-w} & x_1^z & -x_0^\lambda \\ x_0^\mu & -x_2^w & x_1^{u-z} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Let e_i denote the $i-th$ unit row vector. By Shaps' algorithm we get the following set of generators of $Hom(I/I^2, B)$ as a B-module:

$$h_{11} : \begin{cases} f_1 \mapsto \det(e_1, e_1, r_2) = & 0 \\ f_2 \mapsto \det(e_1, e_2, r_2) = & x_1^{u-z} \\ f_3 \mapsto \det(e_1, e_3, r_2) = & x_2^w \end{cases} \qquad h_{12} : \begin{cases} f_1 \mapsto \det(e_1, r_1, e_1) = & 0 \\ f_2 \mapsto \det(e_1, r_1, e_2) = & x_0^\lambda \\ f_3 \mapsto \det(e_1, r_1, e_3) = & x_1^z \end{cases}$$

$$h_{21} : \begin{cases} f_1 \mapsto \det(e_2, e_1, r_2) = & -x_1^{u-z} \\ f_2 \mapsto \det(e_1, e_2, r_2) = & 0 \\ f_3 \mapsto \det(e_1, e_3, r_2) = & x_0^\mu \end{cases} \qquad h_{22} : \begin{cases} f_1 \mapsto \det(e_2, r_1, e_1) = & -x_0^\lambda \\ f_2 \mapsto \det(e_2, r_1, e_2) = & 0 \\ f_3 \mapsto \det(e_2, r_1, e_3) = & x_2^{v-w} \end{cases}$$

$$h_{31}:\left\{\begin{array}{ll} f_1\mapsto \text{det}(e_3,e_1,r_2)= & -x_2^w\\ f_2\mapsto \text{det}(e_2,e_2,r_2)= & -x_0^{\mu}\\ f_3\mapsto \text{det}(e_1,e_3,r_2)= & 0\end{array}\right. \qquad h_{32}:\left\{\begin{array}{ll} f_1\mapsto \text{det}(e_3,r_1,e_1)= & -x_1^z\\ f_2\mapsto \text{det}(e_3,r_1,e_2)= & -x_2^{v-w}\\ f_3\mapsto \text{det}(e_3,r_1,e_3)= & 0\end{array}\right.$$

We can construct the infinitesimal deformation (not miniversal, since dim T^1_B is greater, in general, but the other generators as vector space have greater degrees).

$$F=f+\epsilon\left[U_1\begin{pmatrix}0\\ x_1^{u-z}\\ x_2^w\end{pmatrix}+U_2\begin{pmatrix}0\\ x_0^{\lambda}\\ x_1^z\end{pmatrix}+U_3\begin{pmatrix}-x_1^{u-z}\\ 0\\ x_0^{\mu}\end{pmatrix}+U_4\begin{pmatrix}-x_0^{\lambda}\\ 0\\ x_2^{v-w}\end{pmatrix}+\right.$$

$$\left.+U_5\begin{pmatrix}-x_2^w\\ -x_0^{\mu}\\ 0\end{pmatrix}+U_6\begin{pmatrix}-x_1^z\\ x_2^{v-w}\\ 0\end{pmatrix}\right]=f+\epsilon g.$$

With: weight$(U_1,...,U_6)=((v-w)n_2,\mu n_0,zn_1,wn_2,\lambda n_0,(u-z)n_1)$.

A relation matrix for F is $R=r+\epsilon\rho,$ with $\rho=\begin{pmatrix}-U_1 & -U_3 & -U_5\\ -U_2 & -U_4 & -U_6\end{pmatrix}.$

In fact

$$rg=U_1\begin{pmatrix}f_1\\ 0\end{pmatrix}+U_2\begin{pmatrix}0\\ f_1\end{pmatrix}+U_3\begin{pmatrix}f_2\\ 0\end{pmatrix}+U_4\begin{pmatrix}0\\ f_2\end{pmatrix}+U_5\begin{pmatrix}f_3\\ 0\end{pmatrix}+U_6\begin{pmatrix}0\\ f_3\end{pmatrix}.$$

Now the equation $(r+\epsilon\rho)(f+\epsilon g+\epsilon^2h)=\epsilon(rg+\rho f)+\epsilon^2(\rho g+rh)=0$
has the solution

$$h=\begin{pmatrix}U_3U_6-U_4U_5\\ U_2U_5-U_1U_6\\ U_1U_4-U_2U_3\end{pmatrix}.$$

Further the entries of h are the 2×2 minors of the matrix ρ so that $\rho h=0$: hence there are no obstructions (conditions on $\{U_i\}$ necessary to have flatness).
The lift to a deformation with parameter space Spec$(\mathbb{F}[U_1,...,U_6])$ is

$$F=f+U_1\begin{pmatrix}0\\ x_1^{u-z}\\ x_2^w\end{pmatrix}+U_2\begin{pmatrix}0\\ x_0^{\lambda}\\ x_1^z\end{pmatrix}+U_3\begin{pmatrix}-x_1^{u-z}\\ 0\\ x_0^{\mu}\end{pmatrix}+U_4\begin{pmatrix}-x_0^{\lambda}\\ 0\\ x_2^{v-w}\end{pmatrix}+$$

$$+U_5\begin{pmatrix}-x_2^w\\ -x_0^{\mu}\\ 0\end{pmatrix}+U_6\begin{pmatrix}-x_1^z\\ x_2^{v-w}\\ 0\end{pmatrix}+\begin{pmatrix}U_3U_6-U_4U_5\\ U_2U_5-U_1U_6\\ U_1U_4-U_2U_3\end{pmatrix}.$$

Since X is smoothable [29], we deduce in particular that $(0,0,0)$ is a regular point on the general fibre: hence

$$1\in\{u-z,z,v-w,w,\lambda,\mu\}.$$

4.2 The example of Buchweitz.

We show what happens in the following case of a non-smoothable monomial curve.

Example 4.3 This example due to Buchweitz [2] shows the first known case of non-smoothable monomial curve (see [2]). We calculate explicitly the miniversal deformation. Let

$$S =< 13, 14, 15, 16, 17, 18, 20, 22, 23 >$$

The ring $B = \mathbb{F}[S]$ has 32 equations in $\mathbb{F}[x_0, \ldots, x_8]$ (found by means of CoCoA [5]):

$$\begin{array}{rrrr}
-x_1^2 + x_0x_2 & -x_2^2 + x_1x_3 & -x_1x_2 + x_0x_3 & -x_3^2 + x_2x_4 \\
-x_2x_3 + x_1x_4 & -x_1x_3 + x_0x_4 & -x_4^2 + x_3x_5 & -x_3x_4 + x_2x_5 \\
-x_2x_4 + x_1x_5 & -x_1x_4 + x_0x_5 & -x_5^2 + x_3x_6 & -x_4x_5 + x_2x_6 \\
-x_3x_5 + x_1x_6 & -x_2x_5 + x_0x_6 & x_0^2x_1 - x_6^2 & -x_0^2x_3 + x_6x_7 \\
-x_6^2 + x_5x_7 & -x_0^3 + x_4x_7 & -x_5x_6 + x_3x_7 & -x_4x_6 + x_2x_7 \\
-x_3x_6 + x_1x_7 & -x_2x_6 + x_0x_7 & x_0^2x_5 - x_7^2 & -x_0x_1x_5 + x_7x_8 \\
-x_0^2x_4 + x_6x_8 & -x_0^2x_2 + x_5x_8 & -x_5x_7 + x_4x_8 & -x_4x_7 + x_3x_8 \\
-x_3x_7 + x_2x_8 & -x_2x_7 + x_1x_8 & -x_1x_7 + x_0x_8 & -x_0^2x_6 + x_8^2
\end{array}$$

The Jacobian matrix whose $rank_P$ is 8 if $P \in X, \;\; P \neq (0, \ldots, 0)$ is the following:

$$
J = \begin{bmatrix}
x_2 & -2x_1 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_3 & -2x_2 & x_1 & 0 & 0 & 0 & 0 & 0 \\
x_3 & -x_2 & -x_1 & x_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_4 & -2x_3 & x_2 & 0 & 0 & 0 & 0 \\
0 & x_4 & -x_3 & -x_2 & x_1 & 0 & 0 & 0 & 0 \\
x_4 & -x_3 & 0 & -x_1 & x_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_5 & -2x_4 & x_3 & 0 & 0 & 0 \\
0 & 0 & x_5 & -x_4 & -x_3 & x_2 & 0 & 0 & 0 \\
0 & x_5 & -x_4 & 0 & -x_2 & x_1 & 0 & 0 & 0 \\
x_5 & -x_4 & 0 & 0 & -x_1 & x_0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_6 & 0 & -2x_5 & x_3 & 0 & 0 \\
0 & 0 & x_6 & 0 & -x_5 & -x_4 & x_2 & 0 & 0 \\
0 & x_6 & 0 & -x_5 & 0 & -x_3 & x_1 & 0 & 0 \\
x_6 & 0 & -x_5 & 0 & 0 & -x_2 & x_0 & 0 & 0 \\
2x_0x_1 & x_0^2 & 0 & 0 & 0 & 0 & -2x_6 & 0 & 0 \\
-2x_0x_3 & 0 & 0 & -x_0^2 & 0 & 0 & x_7 & x_6 & 0 \\
0 & 0 & 0 & 0 & 0 & x_7 & -2x_6 & x_5 & 0 \\
-3x_0^2 & 0 & 0 & 0 & x_7 & 0 & 0 & x_4 & 0 \\
0 & 0 & 0 & x_7 & 0 & -x_6 & -x_5 & x_3 & 0 \\
0 & 0 & x_7 & 0 & -x_6 & 0 & -x_4 & x_2 & 0 \\
0 & x_7 & 0 & -x_6 & 0 & 0 & -x_3 & x_1 & 0 \\
x_7 & 0 & -x_6 & 0 & 0 & 0 & -x_2 & x_0 & 0 \\
2x_0x_5 & 0 & 0 & 0 & 0 & x_0^2 & 0 & -2x_7 & 0 \\
-x_1x_5 & -x_0x_5 & 0 & 0 & 0 & -x_0x_1 & 0 & x_8 & x_7 \\
-2x_0x_4 & 0 & 0 & 0 & -x_0^2 & 0 & x_8 & 0 & x_6 \\
-2x_0x_2 & 0 & -x_0^2 & 0 & 0 & x_8 & 0 & 0 & x_5 \\
0 & 0 & 0 & 0 & x_8 & -x_7 & 0 & -x_5 & x_4 \\
0 & 0 & 0 & x_8 & -x_7 & 0 & 0 & -x_4 & x_3 \\
0 & 0 & x_8 & -x_7 & 0 & 0 & 0 & -x_3 & x_2 \\
0 & x_8 & -x_7 & 0 & 0 & 0 & 0 & -x_2 & x_1 \\
x_8 & -x_7 & 0 & 0 & 0 & 0 & 0 & -x_1 & x_0 \\
-2x_0x_6 & 0 & 0 & 0 & 0 & 0 & -x_0^2 & 0 & 2x_8
\end{bmatrix}
$$

Now we summarize the computation of $\dim T^1_B(\ell)$ by means of the formula

$$\dim T^1_B(\ell) = \#G_\ell - 1 - \rho_\ell.$$

It is useful to consider the Jacobian matrix evaluated in $P(1, \ldots, 1)$ with the rows ordered by degree: here the first column shows the weighted degrees of the equations.

$$
\left(\text{deg} \mid J(1)\right) =
\begin{array}{c|rrrrrrrrr}
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\hline
\text{deg} & 13 & 14 & 15 & 16 & 17 & 18 & 20 & 22 & 23 \\
\hline\hline
28 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
29 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
30 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
31 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
31 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
32 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
32 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
33 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\
33 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
34 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
34 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
35 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\
35 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\
36 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\
36 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\
36 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
37 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
37 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 \\
38 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\
38 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\
39 & -3 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
39 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
40 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
40 & 2 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
40 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
41 & -2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
42 & -2 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
43 & -2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
44 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\
45 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
46 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2
\end{array}
$$

The matrix associated to degree 0 derivations mod I is

$$
J_0 = \left(x_i \frac{\partial F_j}{\partial x_i} \right) = \begin{bmatrix} t^{28} & 0 & \dots & 0 \\ 0 & t^{29} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & t^{46} \end{bmatrix} J(1).
$$

Now we show that $\dim_{\mathbf{F}} T^1_B = 21$; to find a basis for $T^1_B(\ell)$ we have to solve the homogeneous system associated to the minor of $J(1)$ formed by the rows of weight $\in H_\ell$.

Step (0) First for each $\ell \in \mathbb{Z}$ we describe the subsets $G_{ell},\ H_{ell}$.

ℓ	G_ℓ	$\#G_\ell$	H	ρ	dim T^1_ℓ
-23	$\{0,..,7\}$	8	$\{28,..,35,42,44\}$	$\rho=7$	0
-22	$\{0,..,6,8\}$	8	$\{28,..,34,41,43,46\}$	$\rho=7$	0
-21	$\{0,...,8\}$	9	$\{28,..,33,40,42,45,46\}$	$\rho=8$	0
-20	$\{0,...,5,7,8\}$	8	$\{28,...,32,39,41,44,45\}$	$\rho=8$	0
-19	$\{0,...,8\}$	9	$\{28,...,31,38,40,43,44\}$	$\rho=8$	0
-18	$\{0,...,4,6,7,8\}$	8	$\{28,...,30,37,39,42,43\}$	$\rho=8$	0
-17	$\{0,...,3,5,...,8\}$	8	$\{28,29,36,38,41,42\}$	$\rho=7$	0
-16	$\{0,1,2,5,6,7,8\}$	7	$\{28,35,37,40,41\}$	$\rho=7$	0
-15	$\{0,1,3,...,8\}$	8	$\{34,36,39,40\}$	$\rho=7$	0
-14	$\{0,2,,...,8\}$	8	$\{33,35,38,39\}$	$\rho=7$	0
-13	$\{1,...,8\}$	8	$\{32,34,37,38\}$	$\rho=7$	0
-12	$\{0,...,8\}$	9	$\{31,33,36,37\}$	$\rho=7$	1
-11	$\{0,...,8\}$	9	$\{30,32,35,36\}$	$\rho=7$	1
-10	$\{0,...,7\}$	8	$\{29,31,34,35\}$	$\rho=6$	1
-9	$\{0,...,6\}$	7	$\{28,30,33,34\}$	$\rho=5$	1
-8	$\{0,...,6\}$	7	$\{29,32,33\}$	$\rho=5$	1
-7	$\{0,...,5\}$	6	$\{28,31,32\}$	$\rho=4$	1
-6	$\{0,...,5\}$	6	$\{30,31\}$	$\rho=4$	1
-5	$\{0,...,4\}$	5	$\{29,30\}$	$\rho=3$	1
-4	$\{0,1,2,3,8\}$	5	$\{28,29\}$	$\rho=2$	2
-3	$\{0,1,2,7\}$	4	$\{28\}$	$\rho=1$	2
-2	$\{0,1,8\}$	3	$\emptyset$		2
-1	$\{0,6,7\}$	3	$\emptyset$		2
1	$\{5,6\}$	2	$\emptyset$		1
2	$\{4\}$	1	$\emptyset$		0
3	$\{3,5\}$	2	$\emptyset$		1
4	$\{2,4\}$	2	$\emptyset$		1
5	$\{1,3\}$	2	$\emptyset$		1
6	$\{0,2\}$	2	$\emptyset$		1

Step (1) By using "FreeMat" (see [11]) we can construct the miniversal deformation (we present in detail the case $\ell=-12$ with $H_\ell=\{31,33,36,37\}$.
Let a be the submatrix of $J(1)$ formed by the rows with degrees $\in H_\ell$:

$$
a = \begin{array}{r||rrrrrrrrr}
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\text{deg} & 13 & 14 & 15 & 16 & 17 & 18 & 20 & 22 & 23 \\
\hline\hline
31 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
31 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
33 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\
33 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
36 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\
36 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\
36 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
37 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
37 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0
\end{array}
$$

Step (1.1)
Write the matrix b obtained by deleting the deg-column in a, find $rank(b)$ and a total reduction c of b, that is

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 9 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -9 & 8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -8 & 7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -7 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step (1.2) Let $\Delta_i := x_i \frac{\partial}{\partial x_i},\ i = 0, \ldots, 8$ (degree 0 derivations). Find the degree-0 derivation whose coefficients are a solution of the homogeneous system associated to c and by using the Euler's identity, we obtain a solution where the coefficient of Δ_0 is null. Then:

$T^1_B(-12) =< t^{-12}\Delta(1) >$, with $\Delta(1) := \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 + 5\Delta_5 + 7\Delta_6 + 9\Delta_7 + 10\Delta_8$.

Step (1.3) Let $e := [0, 1, 2, 3, 4, 5, 7, 9, 10]^T$, to obtain the image g_1 of $\Delta(1)$ make the product: $J(1)e = \left[\ 0, 0, ..., 0, 13, 0, 0, -13, 0, 13, 13, 13, -13, 13, 13\ \right]^T$ and so $\Delta(1)$ takes f to $g_1 =$ $[t^{28}, t^{29}, t^{30}, t^{30}, t^{31}, t^{31}, t^{32}, t^{32}, t^{33}, t^{33}, t^{34}, t^{34}, t^{35}, t^{35}, t^{36}, t^{36}, t^{36}, t^{37}, t^{37}, t^{38}, t^{38}, t^{39}, t^{39}, t^{40},$ $t^{40}, t^{40}, t^{41}, t^{42}, t^{43}, t^{44}, t^{45}, t^{46}] * J(1)e =$
$\left[\ 0, 0, ...0, 13t^{27}, 0, 0, -13t^{28}, 0, 13t^{29}, 13t^{30}, 13t^{31}, -13t^{32}, 13t^{33}, 13t^{34}\ \right]^T =$
$\left[\ 0, 0, ...0, 13x_0x_1, 0, 0, -13x_1^2, 0, 13x_1x_2, 13x_2^2, 13x_1x_4, -13x_1x_5, 13x_2x_5, 13x_4^2\ \right]^T \in \left(M^2\right)^{32}$.
(Here $*$ denotes the $pairwise$ vector product). Analogously we have:
$T^1_B(-11) = t^{-11} < \Delta(1) >$.
$f \mapsto g_2 = \left[\ 0, 0, ...0, 13t^{28}, 0, 0, -13t^{29}, 0, 13t^{30}, 13t^{31}, 13t^{32}, -13t^{33}, 13t^{34}, 13t^{35}\ \right]^T =$
$\left[\ 0, 0, ...0, 13x_1^2, 0, 0, -13x_1x_2, 0, 13x_2^2, 13x_1x_4, 13x_1x_5, -13x_2x_5, 13x_4^2, 13x_4x_5\ \right]^T \in \left(M^2\right)^{32}$.
$T^1_B(-10) = t^{-10}\Delta(2)\ with\ \Delta(2) := \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 + 5\Delta_5 + 7\Delta_6 + 9\Delta_7$.

With image $g_3 = t^{-10}[t^{28}, t^{29}, t^{30}, t^{30}, t^{31}, t^{31}, t^{32}, t^{32}, t^{33}, t^{33}, t^{34}, t^{34}, t^{35}, t^{35}, t^{36}, t^{36}, t^{36}, t^{37},$ $t^{37}, t^{38}, t^{38}, t^{39}, t^{39}, t^{40}, t^{40}, t^{40}, t^{41}, t^{42}, t^{43}, t^{44}, t^{45}, t^{46}] * [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$ -10,-10, 0,-10, 0, 13,-10, 0,-13,-10, 3, 13, 3,-13, 3,-7$] = [0, \ldots, 0,$-10t^{26},-10$t^{27}, 0, \ldots]$.
Therefore $f \mapsto g_3 \in \left(M^2\right)^{32}$. In the same way one obtains:
$T^1_B(-9) = t^{-10} < \Delta(3) >,$ with $\Delta(3) = \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 + 5\Delta_5 + 7\Delta_6;$
$T^1_B(-8) = t^{-8} < \Delta(3) >$. Further one can find that
$T^1_B(-7) = t^{-7} < \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 + 5\Delta_5 >$
$T^1_B(-6) = t^{-6} < \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 + 5\Delta_5 >$
$T^1_B(-5) = t^{-5} < \Delta_1 + 2\Delta_2 + 3\Delta_3 + 4\Delta_4 >$
$T^1_B(-4) = t^{-4} < \Delta_1 + 2\Delta_2 + 3\Delta_3,\ \Delta_8 >$
$T^1_B(-3) = t^{-3} < \Delta_1 + 2\Delta_2,\ \Delta_7 >$: $T^1_B(-2) = t^{-2} < \Delta_1,\ \Delta_8 >$
$T^1_B(-1) = t^{-1} < \Delta_6,\ \Delta_7 >$
$T^1_B(1) = t < \Delta_5 >$
$T^1_B(3) = t^3 < \Delta_5 >$

$T_B^1(4) = t^4 < \Delta_4 >$
$T_B^1(5) = t^5 < \Delta_3 >$
$T_B^1(6) = t^6 < \Delta_2 >$.

We conclude that each generator of T_B^1 sends $f \mapsto g \in \left(M^2\right)^{32}$.
Hence all the hypersurfaces defined by the equations $F_i \in \mathbb{F}[x_0, \ldots, x_8, \alpha_1, \ldots, \alpha_{21}],\ i = 1 \ldots, 32$ of the miniversal deformation are singular at $P(0, 0, \ldots, 0, \alpha_1, \ldots, \alpha_{21})$. In particular every fibre $X_{\underline{T}}$ is singular: this means that X is non-smoothable.

5 Arithmetic sequences of embedding dimension four.

In this section using the above algorithms we prove that semigroups of embedding dimension four generated by an arithmetic sequence are Weierstrass.

First we recall how to find the generators for the ideal I of the monomial curve associated to the semigroup. We refer to the paper [24] and we use the same notations.

Notation 5.1 Assume $S =< n_0, \ldots, n_p, n_{p+1} >$, with $n_i = n_0 + id$ (minimal system of generators), and denote by $Ap(S)$ the Apery set respect to n_0.
Let $a, b \in \mathbb{N}$ such that $n_0 = a(p + 1) + b$, with $a \geq 1,\ 0 \leq b \leq p$.
For each $t \in \mathbb{N}$, let $q_t,\ r_t$ with $1 \leq r_t \leq p$ such that $t = q_t p + r_t$, let $g_t := q_t n_p + n_{r_t}$ and let

$$\begin{cases} u := \min\{t \in \mathbb{N} \mid g_t \notin Ap(S)\} \\ v := \min\{n \in \mathbb{N} \mid vn_{p+1} \in < n_0, ..., n_p >\} \end{cases}$$

$$\text{Then} \begin{cases} g_u = \lambda n_0 + w n_{p+1},\ \lambda \geq 1 \\ v n_{p+1} = \mu n_0 + g_z,\ \ v \geq 2,\ \ v > w,\ 0 \leq z < u \\ \quad \text{further :} \\ (\lambda + \mu) n_0 = g_{u-z} + (v - w) n_{p+1}. \end{cases} \qquad (*)$$

It is easy to see that $u = p + 1,\ \lambda = w = 1$ and
if $b = 0$, then $z = 0$.
If $b \geq 1 : v = a + 1,\ \ \mu = a + d \geq 2,\ \ z = p + 1 - b$
and a minimal set of generators for the ideal I is the union of the following sets:

$$\xi_{ij} = \left[\begin{array}{lll} x_i x_j - x_0 x_{i+j} & \text{if} & i + j \leq p,\ \ 1 \leq i \leq j \\ x_i x_j - x_{i+j-p} x_p & \text{if} & i + j > p,\ \ 1 \leq i \leq j \leq p - 1 \end{array} \right.$$

$$\phi_i = x_{1+i} x_p - x_i x_{p+1} \ \text{ with } \ 0 \leq i \leq p - 1$$

$$\psi_j = x_{b+j} x_{p+1}^{v-1} - x_0^{\mu} x_j \ \text{ with } \ 0 \leq j \leq p - b$$

$$\theta = x_{p+1}^{v} - x_0^{\mu} x_{p+1-b}.$$

Now we deal with the case $p = 2$ (embdim$(S) = 4$): here

$$\{\xi_{ij}\} = \{\xi_{11}\} = \{x_1^2 - x_0 x_2\}, \ \text{ with } \deg(\xi_{11}) = 2n_1,$$

$$\{\phi_i\} = \{\phi_0, \phi_1\} = \{x_1 x_2 - x_0 x_3,\ x_2^2 - x_1 x_3\}, \ \deg(\phi_0) = n_1 + n_2,\ \deg(\phi_1) = 2n_2,$$

$$\{\psi_j\} = \left[\begin{array}{lll} \{\psi_0, \psi_1\} = \{x_1 x_3^{v-1} - x_0^{1+\mu},\ \ x_2 x_3^{v-1} - x_0^{\mu} x_1\} & \text{if} & b = 1,\ \deg(\psi_1) = \mu n_0 + n_1, \\ \{\psi_0\} = \{x_2 x_3^{v-1} - x_0^{1+\mu}\} & \text{if} & b = 2,\ \deg(\psi_0) = (1 + \mu) n_0, \end{array} \right.$$

$$\theta = \left[\begin{array}{lll} x_3^v - x_0^\mu x_{p+1-b}, & \deg(\theta) = \mu n_0 + n_{p+1-b} & \text{if } \ b = 1, 2 \\ x_3^v - x_0^\mu, & \deg(\theta) = \mu n_0 & \text{if } \ b = 0 \end{array} \right. .$$

Hence the equations for the associated monomial curve in $\mathbb{A}^4$ are :

$$(b=0) \begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_3^v - x_0^\mu \end{pmatrix}; (b=1) \begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_1x_3^{v-1} - x_0^{1+\mu} \\ x_2x_3^{v-1} - x_0^\mu x_1 \\ x_3^v - x_0^\mu x_2 \end{pmatrix}; (b=2) \begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_2x_3^{v-1} - x_0^{1+\mu} \\ x_3^v - x_0^\mu x_1 \end{pmatrix}.$$

Lemma 5.2 *Assume $S =< n_0, n_1, n_2, n_3 >$, minimally generated by an arithmetic sequence. With notation fixed in (5.1) we have:*

1. $T_B^1(-\mu n_0) =< t^{-\mu n_0}(\Delta_1 + 2\Delta_2 + 3\Delta_3) >$.
2. *Further in case $b = 2$, we have*

$$T_B^1(-(v-1)n_3) =< t^{-(v-1)n_3}(\Delta_1 + 2\Delta_2 + 3\Delta_3) >$$

$$T_B^1(-n_2) =< t^{-n_2}(2v\Delta_1 + (v+1)\Delta_2 + 2\Delta_3) >$$

Proof. (1). With notations (3.14) and (5.1),assume $\ell = -\mu n_0$. Further recall that $\mu \geq 2$. Hence $\#G_\ell = 4$. Now proceed separately according that $b = 0, 1, 2$.

(**Case** $b = 0$). Easily one can see that $H_\ell = \{2n_1, n_1 + n_2, 2n_2\}$.
The degree 0 Jacobian matrix in this case is

$$J(0) = \left(x_i \frac{\partial f_j}{\partial x_i} \right) = \begin{pmatrix} -x_0x_2 & 2x_1^2 & -x_0x_2 & 0 \\ -x_0x_3 & x_1x_2 & x_1x_2 & -x_0x_3 \\ 0 & -x_1x_3 & 2x_2^2 & -x_1x_3 \\ -\mu x_0^\mu & 0 & 0 & vx_3^v \end{pmatrix}.$$

The evaluation of this matrix in $P(1, \ldots, 1) \in X$ is

$$J(1) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ -\mu & 0 & 0 & v \end{pmatrix}$$

Then $\dim(V_\ell) = 2$ and $\dim(T_B^1(-\mu n_0)) = 4 - 2 - 1 = 1$. A vector $(0, a, b, c)^T$ such that $J(1)(0, a, b, c)^T$ has the first three entries null is $(0, 1, 2, 3)^T$. We obtain that a basis of

$$T_B^1(-\mu n_0) \ \text{ is } \ t^{-\mu n_0}(\Delta_1 + 2\Delta_2 + 3\Delta_3).$$

(**Case** $b = 1$). We have $H_\ell = \left[\begin{array}{ll} \{2n_1, n_1 + n_2\} & if \ v = 2 \\ \{2n_1, n_1 + n_2, 2n_2\} & if \ v > 2 \end{array} \right.$. In fact
$2n_1 - \mu n_0 = n_2 + (1 - \mu)n_0 \notin S$ since $\mu \geq 2$ and $\{n_i\}$ is a minimal set of generators,
$n_1 + n_2 - \mu n_0 = 3d - (\mu - 2)n_0 \notin S$, since $n_0 + 3d = n_3$,

$$2n_2 - \mu n_0 = 3n_2 - vn_3 = (3 - v)n_0 + (6 - 3v)d = \left[\begin{array}{rl} n_0, & if \ v = 2 \\ < n_0, & if \ v > 2 \end{array} \right.$$

for any other generator f_j of the ideal I, obviously deg$(f_j) - \mu n_0 \in S$.
The deg 0 Jacobian matrix is

$$J(0) = \left(x_i \frac{\partial f_j}{\partial x_i}\right) = \begin{pmatrix} -x_0x_2 & 2x_1^2 & -x_0x_2 & 0 \\ -x_0x_3 & x_1x_2 & x_1x_2 & -x_0x_3 \\ 0 & -x_1x_3 & 2x_2^2 & -x_1x_3 \\ -(1+\mu)x_0^{1+\mu} & x_1x_3^{v-1} & 0 & (v-1)x_1x_3^{v-1} \\ -\mu x_0^{\mu}x_1 & -x_0^{\mu}x_1 & x_2x_3^{v-1} & (v-1)x_2x_3^{v-1} \\ -\mu x_0^{\mu}x_2 & 0 & -x_0^{\mu}x_2 & vx_3^{v} \end{pmatrix}$$

The evaluation of this matrix in $P(1,\dots,1) \in X$ is

$$J(1) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ -(1+\mu) & 1 & 0 & (v-1) \\ -\mu & -1 & 1 & (v-1) \\ -\mu & 0 & -1 & v \end{pmatrix}$$

In both cases we see that dim$(V_\ell) = 2$. Then dim$(T_B^1(-\mu n_0)) = 4 - 2 - 1 = 1$ and analogously to case $b = 0$, we recover the same basis for $T_B^1(-\mu n_0)$.

(**Case** $b = 2$) As above: $H_\ell = \{2n_1, n_1 + n_2, 2n_2\}$ because
$2n_2 - \mu n_0 = n_1 + 2n_2 - vn_3 = (3 - v)n_0 + (5 - 3v)d \notin S$ (it is $< n_0$).

The degree 0 Jacobian matrix in this case is

$$J(0) = \begin{pmatrix} -x_0x_2 & 2x_1^2 & -x_0x_2 & 0 \\ -x_0x_3 & x_1x_2 & x_1x_2 & -x_0x_3 \\ 0 & -x_1x_3 & 2x_2^2 & -x_1x_3 \\ -(1+\mu)x_0^{1+\mu} & 0 & x_2x_3^{v-1} & (v-1)x_2x_3^{v-1} \\ -\mu x_0^{\mu}x_1 & -x_0^{\mu}x_1 & 0 & vx_3^{v} \end{pmatrix}$$

this matrix evaluated in $P(1,\dots,1)$ is

$$J(1) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ -(1+\mu) & 0 & 1 & (v-1) \\ -\mu & -1 & 0 & v \end{pmatrix}$$

Therefore dim$(V_\ell) = 1$, a basis for $T_B^1(-\mu n_0)$ is $t^{-\mu n_0}(\Delta_1 + 2\Delta_2 + 3\Delta_3)$.

(3). Let $\ell = -(v-1)n_3$. Then:

$$G_\ell = \left[\begin{array}{ll} \{0,1,2\}, & if\ v = 2 \\ \{0,1,2,3\}, & if\ v > 2 \end{array} \right. , \quad H_\ell = \left[\begin{array}{ll} \{2n_1\} & if\ v = 2 \\ \{2n_1, n_1+n_2, 2n_2\} & if\ v > 2 \end{array} \right. .$$

In fact: $2n_1 - (v-1)n_3 \leq 2n_1 - n_3 = n_0 - d \notin S$
$n_1 + n_2 - (v-1)n_3 = n_0 \in S$, if $v = 2$, $n_1 + n_2 - (v-1)n_3 < 0$, if $v > 2$.
$2n_2 - (v-1)n_3 = n_1 \in S$, if $v = 2$, $n_1 + n_3 - (v-1)n_3 < 0$, if $v > 2$.
for any other generator f_j of the ideal I , obviously $\deg(f_j) - \mu n_0 \in S$.
In both cases we conclude that $\dim(T^1_B(\ell)) = 1$, with basis

$$t^{-(v-1)n_3}(\Delta_1 + 2\Delta_2 + 3\Delta_3).$$

Let now $\ell = -n_2$. Then: $\quad G_\ell = \{0, 1, 3\}, \; H_\ell = \{vn_3\}$.
In fact assume $vn_3 - n_2 \in S$, i.e., $vn_3 = \alpha n_0 + \beta n_1 + \gamma n_2 + \delta n_3$, with $\gamma \geq 1$, then $\delta = 0$ by the minimality of v; $\beta \geq 1 \Longrightarrow$ (since $vn_3 = \mu n_0 + n_1$) $\quad \mu n_0 + n_1 = \alpha n_0 + (\beta-1)n_1 + (\gamma-1)n_2 + n_0 + n_3 \Longrightarrow (v-1)n_3 \in < n_0, n_1, n_2 >$, contradiction. Then $\beta = 0$ and so $vn_3 = \alpha n_0 + \gamma n_2$, impossible since the residues mod n_0 cannot be equal. We conclude that $\dim(T^1_B(\ell) = 1$ and a basis is $t^{-n_2}(2v\Delta_1 + (v+1)\Delta_2 + 2\Delta_3)$.

In next theorem we prove that any semigroup generated by an arithmetic sequence with embedding dimension 4 is Weierstrass. Further we find the equations of a 1-parameter flat family of smooth projective with only one point P_∞ at infinity and the semigroup associated at P_∞ equal to S. This is done by using Pinkham's algorithm [25].

Theorem 5.3 *With notation 5.2, assume the semigroup $S =< n_0, n_1, n_2, n_3 >$ minimally generated by an arithmetic sequence and let $X :=$ Spec$(\mathbb{F}[S])$: then X is smoothable and S is Weierstrass. More precisely there exists one deformation Y of X with smooth generic fibres (projective curves) and parameter space $\mathbb{A}^1_{\mathbb{F}}$:*

1. *If* $b = 0$, *the equations* $F = f + Ux_4^{\mu n_0}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_3^v - x_0^\mu + Ux_4^{\mu n_0} \end{pmatrix}$

   define the required deformation $\pi : Y \longrightarrow \mathbb{A}^1_{\mathbb{F}}$.

2. *If* $b = 1$, *the equations* $F = f + Ux_4^{\mu n_0}\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_1x_3^{v-1} - x_0^{1+\mu} + Ux_0x_4^{\mu n_0} \\ x_2x_3^{v-1} - x_0^\mu x_1 + Ux_1x_4^{\mu n_0} \\ x_3^v - x_0^\mu x_2 + Ux_2x_4^{\mu n_0} \end{pmatrix}$

   define the required deformation $\pi : Y \longrightarrow \mathbb{A}^1_{\mathbb{F}}$.

3. *If* $b=2$, $F=$

$$F=\begin{pmatrix} x_1^2-x_0x_2+Ux_0x_4^{n_2} \\ x_1x_2-x_0x_3+Ux_1x_4^{n_2} \\ x_2^2-x_1x_3-U^2x_4^{2n_2} \\ x_2x_3^{v-1}-x_0^{1+\mu}+Ux_2x_4^{(v-1)n_3}+Ux_3^{v-1}x_4^{n_2}+U^2x_4^{(v-1)n_3+n_2} \\ x_3^v-x_0^\mu x_1+Ux_3x_4^{(v-1)n_3} \end{pmatrix}=$$

$$=f+U\begin{pmatrix} x_0x_4^{n_2} \\ x_1x_4^{n_2} \\ 0 \\ x_2x_4^{(v-1)n_3}+x_3^{v-1}x_4^{n_2} \\ x_3x_4^{(v-1)n_3} \end{pmatrix}+U^2\begin{pmatrix} 0 \\ 0 \\ x_4^{2n_2} \\ x_4^{(v-1)n_3+n_2} \\ 0 \end{pmatrix}$$

define the required deformation $\pi : Y \longrightarrow \mathbb{A}^1_{\mathbb{F}}$.

Proof. (**Case** $b=0$). In this case the image of the element found in 5.2 for $T^1_B(-\mu n_0)$ (eigenvector) and the relation matrix among the generators of the ideal are: $g_1=(0,0,0,1)^T$,

$$r=\begin{pmatrix} x_2 & -x_1 & x_0 & 0 \\ -x_3 & x_2 & -x_1 & 0 \\ x_0^\mu-x_3^v & 0 & 0 & x_1^2-x_0x_2 \\ 0 & x_0^\mu-x_3^v & 0 & x_1x_2-x_0x_3 \\ 0 & 0 & x_0^\mu-x_3^v & x_2^2-x_1x_3 \end{pmatrix}=\begin{pmatrix} x_2 & -x_1 & x_0 & 0 \\ -x_3 & x_2 & -x_1 & 0 \\ -f_4 & 0 & 0 & f_1 \\ 0 & -f_4 & 0 & f_2 \\ 0 & 0 & -f_4 & f_3 \end{pmatrix}.$$

Hence an infinitesimal deformation of X is given by the equations $F=f+\varepsilon Ug_1$.
By a direct computation one has

$$rUg_1=U\begin{pmatrix} 0 \\ 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \rho=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -U & 0 & 0 & 0 \\ 0 & -U & 0 & 0 \\ 0 & 0 & -U & 0 \end{pmatrix}, \quad \rho g_1=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Following the Pinkham's method [25, (1.16)], we consider the weighted homogeneous projective space $Proj(\mathbb{F}[x_0,\dots,x_4])$, weight$(x_i)=n_i$, for $0\le i\le 3$, weight$(x_4)=1$) and substitute the variable U with $Ux_4^{\mu n_0}$, (U parameter); therefore we get the deformation Y with parameter space $S=\mathbb{A}^1$ and fibres which are projective curves with only one (regular) point at infinity $P_\infty=(t^{n_0}:\dots:t^{n_3}:0)$, $t\neq 0$. The equations are

$$F=f+Ug_1x_4^{\mu n_0}.$$

To verify that the fibres Y_U, $U\neq 0$ of the family are non-singular curves it suffices to put $x_4=1$ and study the rank of the jacobian matrix of the affine curve $Y_U\cap(x_4\neq 0)$. Now this matrix is equal to the jacobian matrix J of the curve X. We already know that $rank_P(J)=3$ if $P\neq(0,\dots,0)$.

$$J=\begin{pmatrix} -x_2 & 2x_1 & -x_0 & 0 \\ -x_3 & x_2 & x_1 & -x_0 \\ 0 & -x_3 & 2x_2 & -x_1 \\ -\mu x_0^{\mu-1} & 0 & 0 & vx_3^{v-1} \end{pmatrix}.$$

Since by the equations of Y_U we get $P \in Y_U \Longrightarrow P \neq (0, \ldots, 0)$, we are done.

(**Case** $b = 1$). In this case the image of the element found in 5.2 for $T^1_B(-\mu n_0)$ (eigenvector) and the relation matrix among the generators of the ideal are

$$g_1 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_0 \\ x_1 \\ x_2 \end{pmatrix}, \quad r = \begin{pmatrix} x_2 & -x_1 & x_0 & 0 & 0 & 0 \\ -x_3 & x_2 & -x_1 & 0 & 0 & 0 \\ x_3^{v-1} & 0 & 0 & -x_1 & x_0 & 0 \\ 0 & x_3^{v-1} & 0 & -x_2 & 0 & x_0 \\ x_0^{\mu} & -x_3^{v-1} & 0 & 0 & x_1 & -x_0 \\ 0 & x_0^{\mu} & -x_3^{v-1} & -x_3 & x_2 & 0 \\ 0 & 0 & -x_3^{v-1} & 0 & x_2 & -x_1 \\ 0 & 0 & x_0^{\mu} & 0 & -x_3 & x_2 \end{pmatrix}.$$

By a direct computation one has

$$rUg_1 = U \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_1 \\ f_2 \\ 0 \\ f_3 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -U & 0 & 0 & 0 & 0 & 0 \\ 0 & -U & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -U & 0 & 0 & 0 \end{pmatrix}, \quad \rho g_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Following the Pinkham's method as above, we substitute the variable U with $Ux_4^{\mu n_0}$, (U parameter); therefore we get the deformation Y with parameter space $S = \mathbb{A}^1$ and fibres which are projective curves with only one regular point at infinity $P_\infty = (t^{n_0} : \cdots : t^{n_3} : 0),\ t \neq 0$.
The equations are

$$F = f + Ug_1x_4^{\mu n_0}.$$

To verify that the fibres $Y_U,\ U \neq 0$ are non-singular curves it suffices to put $x_4 = 1$ and study the rank of the jacobian matrix of the affine curve $Y_U \cap (x_4 \neq 0)$. This matrix is

$$J = \begin{pmatrix} -x_2 & 2x_1 & -x_0 & 0 \\ -x_3 & x_2 & x_1 & -x_0 \\ 0 & -x_3 & 2x_2 & -x_1 \\ -(1+\mu)x_0^{\mu} + U & x_3^{v-1} & 0 & (v-1)x_1x_3^{v-2} \\ -\mu x_0^{\mu-1}x_1 & -x_0^{\mu} + U & x_3^{v-1} & (v-1)x_2x_3^{v-2} \\ -\mu x_0^{\mu-1}x_2 & 0 & -x_0^{\mu} + U & vx_3^{v-1} \end{pmatrix}.$$

We claim that the rank of $J = 3$ if $U \neq 0$ hence by the jacobian criterion of regularity we deduce that the fibres ar smooth for $U \neq 0$, i.e., the curve X is smoothable and the semigroup is Weierstrass.

In $P_0(0 : \cdots : 0, 1),\ U \neq 0$ we have the non-null minor $\quad \det \begin{pmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix}.$

If $P \neq P_0$ (P belonging to the fibre Y_U of the canonical projection $\pi : Y \longrightarrow \mathbb{A}^1$), according to the equations of Y_U and since $v \geq 2,$ we have

$$\det \begin{pmatrix} 2x_1 & -x_0 & 0 \\ x_2 & x_1 & -x_0 \\ x_3^{v-1} & 0 & (v-1)x_1x_3^{v-2} \end{pmatrix} = x_3^{v-1}x_0^2 + (v-1)x_1x_3^{v-2}(2x_1^2 + x_0x_2) =$$

$= x_3^{v-2}x_0[x_0x_3 + 3(v-1)x_1x_2] = (3v-2)x_0^2x_3^{v-2}.$
If $x_0x_3 = 0$, by the equations we get only P_0. We are done.

(**Case** $b = 2$). In this case one can easily see that the generator found in 5.2 gives a deformation with all singular fibres. Then we need to find a different suitable deformation Y such that the rank of the jacobian matrix is "generically" equal to 3 by the Jacobian criterion of regularity. We claim that a deformation which verifies this condition is

$$F = f + Ug_1 + Vg_2 + h \qquad (*)$$

where g_1, g_2 are the images of the basis of $T_B^1(-(v-1)n_3)$ (resp $T_B^1(-n_2)$) of (5.2.3) and h is found by the flatness conditions. Precisely, the relation matrix r among the generators of the ideal and the vectors g_1, g_2, h are

$$r = \begin{pmatrix} x_2 & -x_1 & x_0 & 0 & 0 \\ -x_3 & x_2 & -x_1 & 0 & 0 \\ 0 & x_3^{v-1} & 0 & -x_1 & x_0 \\ x_0^{\mu} & 0 & x_3^{v-1} & -x_2 & x_1 \\ 0 & x_0^{\mu} & 0 & -x_3 & x_2 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2 \\ x_3 \end{pmatrix}, \quad g_2 = \begin{pmatrix} x_0 \\ x_1 \\ 0 \\ x_3^{v-1} \\ 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ 0 \\ -V^2 \\ UV \\ 0 \end{pmatrix}.$$

To find h, by a direct computation we get

$$rUg_1 = \begin{pmatrix} 0 \\ 0 \\ -f_2 \\ -f_3 \\ 0 \end{pmatrix} = -\rho_1 f, \;\text{ with }\; \rho_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$rVg_2 = V\begin{pmatrix} -f_1 \\ f_2 \\ 0 \\ -f_4 \\ -f_5 \end{pmatrix} = -\rho_2 f, \;\text{ with }\; \rho_2 = \begin{pmatrix} V & 0 & 0 & 0 & 0 \\ 0 & -V & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V & 0 \\ 0 & 0 & 0 & 0 & V \end{pmatrix},$$

Then $\rho = \begin{pmatrix} V & 0 & 0 & 0 & 0 \\ 0 & -V & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & U & V & 0 \\ 0 & 0 & 0 & 0 & V \end{pmatrix}, \quad \rho g = \begin{pmatrix} V^2x_0 \\ -V^2x_1 \\ x_1UV \\ x_2UV + V^2x_3^{v-1} \\ x_3UV \end{pmatrix}.$

Further $\rho g = -rh$, with $h = \begin{pmatrix} 0 \\ 0 \\ -V^2 \\ UV \\ 0 \end{pmatrix}$, finally $\rho h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Following the Pinkham's method as above, we substitute the variables U, V respectively with $Ux_4^{(v-1)n_3}, Vx_4^{n_2}$; therefore we get the deformation Y' with parameter space $S = \mathbb{A}^2$ and fibres with only one (regular) point at infinity. The equations are

$$\begin{pmatrix} x_1^2 - x_0x_2 + Vx_0x_4^{n_2} \\ x_1x_2 - x_0x_3 + Vx_1x_4^{n_2} \\ x_2^2 - x_1x_3 - V^2x_4^{2(n_2)} \\ x_2x_3^{v-1} - x_0^{1+\mu} + Ux_2x_4^{(v-1)n_3} + Vx_3^{v-1}x_4^{n_2} + UVx_4^{(v-1)n_3+n_2} \\ x_3^v - x_0^\mu x_1 + Ux_3x_4^{(v-1)n_3} \end{pmatrix}.$$

Finally we claim that the restriction to the line $(U = V) \subseteq S$ gives a 1-parameter deformation Y with smooth generic fibre. Since the point P_∞ is non-singular, we can put $x_4 = 1$ and study the rank of jacobian matrix of the affine curve $Y \cap (x_4 \neq 0)$:

$$J(Y) = \begin{pmatrix} -x_2 + U & 2x_1 & -x_0 & 0 \\ -x_3 & x_2 + U & x_1 & -x_0 \\ 0 & -x_3 & 2x_2 & -x_1 \\ -(1+\mu)x_0^\mu & 0 & x_3^{v-1} + U & (v-1)x_3^{v-2}(x_2 + U) \\ -\mu x_0^{\mu-1}x_1 & -x_0^\mu & 0 & vx_3^{v-1} + U \end{pmatrix}.$$

As in case $b = 1$ we can assume $P \neq P_0 = (0 : \cdots : 0 : 1)$, P belonging to the fibre Y_U. Consider the minor

$$\det\begin{pmatrix} 2x_1 & -x_0 & 0 \\ x_2 + U & x_1 & -x_0 \\ -x_3 & 2x_2 & -x_1 \end{pmatrix} = \det\begin{pmatrix} 2x_1 & -x_0 & 0 \\ x_2 & x_1 & -x_0 \\ -x_3 & 2x_2 & -x_1 \end{pmatrix} + \det\begin{pmatrix} 2x_1 & -x_0 & 0 \\ U & x_1 & -x_0 \\ -x_3 & 2x_2 & -x_1 \end{pmatrix} =$$

$= -Ux_0x_1$.

If $x_0 = 0$ by the equations we get $x_1 = 0$, $x_3^{v-1} + U = 0$ and $x_2 = +U$, and so by the fourth equation give $Ux_3^{v-1} = 0$, impossible.
If $x_1 = 0$ by the equations we get $x_2 = U$ (since $x_0 \neq 0$ by above), $x_3 = 0$.
The fourth equation gives $-x_0^{\mu+1} + U^2 = 0$.
Now the jacobian matrix in these points is

$$J(Y) = \begin{pmatrix} 0 & 0 & -x_0 & 0 \\ 0 & 2U & 0 & -x_0 \\ 0 & 0 & 2U & 0 \\ -(1+\mu)x_0^\mu & 0 & U & (v-1)x_3^{v-2}(x_2 + U) \\ 0 & -x_0^\mu & 0 & U \end{pmatrix}.$$

Recalling the fourth equation we see that the minor

$$\det\begin{pmatrix} 2U & 0 & -x_0 \\ 0 & 2U & 0 \\ -x_0^\mu & 0 & U \end{pmatrix} = 2U(2U^2 - x_0^{\mu+1}) = 2U^3$$

Hence $\text{rank}(J(Y)) = 3$ for each $P \in Y_U, \ \forall\, U \neq 0$. We are done.

Remark 5.4 Note that in case $b = 2, v > 2$ the deformation Y' of the curve X with equations

$$\begin{pmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \\ x_1x_3^{v-1} - x_0^{1+\mu} + Vx_0 \\ x_3^v - x_0^{\mu}x_1 + Vx_1 \end{pmatrix}$$

has parameter space $\mathbb{A}^1_{\mathbb{F}}$, but every fibre of this deformation has a singularity at the origin: hence in general this construction does not give informations on the smoothability of the curve X even if the algorithm to construct Y' starting from the infinitesimal deformation ends at the first step.

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